Matrix Approach to Cooperative Game Theory

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MATRIX APPROACH TO COOPERATIVE GAME THEORY

PROEFSCHRIFT

ter verkrijging van

de graad van doctor aan de Universiteit Twente,

op gezag van de rector magnificus,

prof. dr. W.H.M. Zijm,

volgens besluit van het College voor Promoties

in het openbaar te verdedigen

op donderdag 28 augustus 2008 om 15:00 uur

door

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geboren op 5 november 1978

te Anhui, China

Dit proefschrift is goedgekeurd door de promotoren

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Contents

Summary

This monograph deals with cooperative games in characteristic function form and solution concepts for these games. Our work focuses on the algebraic representation and the matrix approach in the framework of cooperative game theory. The organization of this work, which consists of six chapters, is as follows.

Chapter 1 is introductory. First of all, the mathematical model of a cooperative game in characteristic function form is described and some examples of cooperative games are given. Secondly, we review the linearity in both the modelling part and the solution part. As a consequence, the algebraic representation and the matrix analysis come forward as a natural and powerful technique in the framework of cooperative game theory.

In Chapter 2, we build the groundwork for applying the algebraic representation and the matrix approach to cooperative game theory. The basic notion of coalitional matrix is introduced. Both a linear game transformation and a linear value on the game space are represented algebraically as the product of the corresponding coalitional matrix and the worth vector. From this, we achieve a matrix approach to study linear transformations as well as linear values by analyzing the structure of these representation coalitional matrices. Four bases for the game space: unity games, unanimity games, dual unanimity games and complementary unanimity games are discussed in terms of three square-coalitional matrices: the Moebius transformation matrix, the dual matrix and the complementary Moebius transformation matrix. The column-coalitional matrices of linear values with some essential properties are described. Particularly, the Shapley standard matrix associated with the

Shapley value is axiomatized.

In Chapter 3, we develop the matrix approach to characterize the Shapley value in terms of two types of associated consistency. Hamiache [37] axiomatized the Shapley value as the unique value satisfying the inessential game property, continuity and associated consistency. The associated game is represented in terms of the associated transformation matrix and the associated consistency for the Shapley value is interpreted as the Shapley standard matrix being invariant under multiplication with the associated transformation matrix. By studying the properties of eigenvalues and eigenvectors of the associated transformation matrix, we use the diagonalization procedure to present an algebraic proof of Hamiache's axiomatization. We introduce the dual similar associated game such that the dual similar associated transformation matrix is similar to the associated transformation matrix in terms of the dual matrix. The similarity of these matrices transforms associated consistency into dual similar associated consistency and vice versa. Together with the inessential game property and continuity, the Shapley value is axiomatized once again.

In Chapter 4, the matrix approach is applied to axiomatizations of linear, symmetric, and efficient values. Driessen [30] extended Hamiache's axiomatization of the Shapley value to this enlarged class of values, which are characterized by the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency. In terms of the $\mathcal{B}$-scaled game, the explicit interrelationships between these values and the Shapley value are exploited. The $\mathcal{B}$-associated transformation matrix of the $\mathcal{B}$-associated game is shown to be similar to the associated matrix in terms of the $\mathcal{B}$-scaling matrix. The adapted $\mathcal{B}$-inessential game property and the $\mathcal{B}$-associated consistency are derived from the similarity of these matrices. By operating $\mathcal{B}$-scaling on the dual similar associated game, we introduce the $\mathcal{B}$-dual similar associated game as well as $\mathcal{B}$-dual similar associated consistency for characterizing this class of values. By analyzing the null space of the Shapley standard matrix, we study the inverse problem of the Shapley value as well as the inverse problem of any linear, symmetric and efficient value.

In Chapter 5, we study the consistency with respect to several types of

reduced games for linear, symmetric and efficient values. We start from the additive efficient normalization of semivalues, which are included in this class. In terms of the explicit interrelationship between the additive efficient normalization of semivalues and the Shapley value, the $\mathcal{B}$-reduced game is defined and the additive efficient normalization of any semivalue is axiomatized through the $\mathcal{B}$-consistency together with covariance and symmetry. As one of our main tools, we introduce the generalization of Sobolev's reduced game. We show that the $\mathcal{B}$-scaled game of the $\mathcal{B}$-reduced game agrees with the generalized Sobolev's reduced game applied to the B-scaled game. By this relationship, each linear, symmetric and efficient value is axiomatized by the $\mathcal{B}$-consistency and the λ-standardness for two-person games. Finally, the relationship between the $\mathcal{B}$-reduced game and the path-independently reduced game is also discussed. Moreover, we extend the $\mathcal{B}$-reduced game (respectively, the linearly reduced game) to the weighted version and show that the weighted $\mathcal{B}$-reduced game (respectively, the weighted linearly reduced game) is equivalent to the standard $\mathcal{B}$-reduced game (respectively, the standard linearly reduced game) for the class of linear, symmetric and efficient values.

Chapter 6 is devoted to the algebraic study of the Harsanyi set and the Weber set in terms of the Moebius transformation and the complementary Moebius transformation. The Harsanyi set of a game is defined as the set of all Harsanyi payoff vectors obtained by distributing the dividend of any coalition S among the players in S for each coalition S. The dividend sharing system, modelled as a column-coalitional matrix M^p, is translated by the Moebius transformation into another sharing system, the second type of a column-coalitional matrix M^q which is associated with the worth vector v. The structure of the worth sharing matrix M^q reflects that any Harsanyi payoff vector is the unique value that satisfies linearity, efficiency, the null player property, and positivity, as axiomatized by Derks, Haller, and Peters [16]. By the inverse of the Moebius transformation, we study the Weber set in terms of the collection of worth sharing matrices $\mathcal{Q}^W$ as well as the collection of dividend sharing matrices $\mathcal{P}^W$, particularly the extreme points of these two collections, which are corresponding to the marginal vectors. We achieve a shorter and intuitive algebraic characterizing procedure for the Weber set by

the Harsanyi payoff vectors. This matrix approach also yields an essential interpretation for the former related results of Vasil'ev [96], Vasil'ev and van der Laan [95], Derks, van der Laan and Vasil'ev [19]. With respect to the complementary dividends, the structure of the corresponding complementary dividend sharing matrix $M^{\bar{p}}$ for the Harsanyi set is also studied in terms of the complementary Moebius transformation. Based on the fact that an extreme point of a linear system is full characterized by its carrier, we construct two suitably chosen linear systems as the second approach for the determination of the extreme points of the Harsanyi set and the Weber set. Moreover, a recursive algorithm for computing the extreme points of the Harsanyi set is presented.

Notation

$\mathbb{N} = \{0, 1, 2, \cdots\}$	the set of natural numbers
$N = \{1, 2, \cdots, n\}$	the player set
$\Omega = \{S \mid S \subseteq N, S \neq \emptyset\}$	the set of all coalitions
s or $\lvert S \rvert$	the cardinality of the set S
$\mathcal{G}^N$	the game space with player set N
$\mathcal{G}$	the universe of all game spaces
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+ = \{a \in R \mid a \geq 0\}$	the set of nonnegative real numbers
$\mathbb{R}^m$	the m-dimensional vector space
$\mathbb{R}^N$	the vector space with coordinates indexed by N
π	the permutation on N
Π^N	the set of all permutations on N
x or $\vec{x}$	the column vector
x'	the transpose of x
$v = (v(S))_{S \in \Omega}$	the worth vector
$\vec{0}$	the column vector with all entries equal to 0
$\mathbf{1}_N, \mathbf{1}_\Omega$	the column vectors with all entries equal to 1
M	the coalitional matrix
M^{-1}	the inverse of M
$\mathbf{0}$	the matrix with all entries equal to 0

Chapter 1

Introduction

1.1 Game theory

Game theory is concerned with studying formal, mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare. The foundations of game theory are laid in the seminal book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern [98], (1944). Game theoretical approaches usually are classified into two branches: noncooperative and cooperative game theory. Noncooperative game theory deals with situations of conflictive and cooperative game theory with situations of cooperation. As such, game theory has important applications to economics (Example 1.2.1) as well as other social sciences (Example 1.2.2).

A conflictive and/or cooperative situation arises naturally when two or more individuals interact. The interaction between the players leads to various payoffs over which each player has his own preferences. Any player tries to obtain his best payoff but the other players may also influence the resulting payoff. The theory of games attempts to put the conflictive and cooperative situations into mathematical models and then to analyze the models. Roughly speaking, the theory of games can be regarded as consisting of two parts, the modelling part and the solution part.

Concerning the modelling part, both conflictive and cooperative situations are described by mathematical models. Most of the models studied in the

1

mathematical theory of games use, more or less, one of the following three abstract forms: the extensive (or tree) form, the normal (or strategic) form and the characteristic function (or coalitional) form. The mathematical models incorporate the rules, the strategic possibilities of the players, the potential payoffs to the players and the preferences of the players over the set of potential payoffs. According to the rules, it is allowed or forbidden that the players communicate with each other and make binding agreements with respect to how they correlate their actions. "Noncooperation" refers to the fact that players cannot make binding agreements, whereas in the "cooperative" framework it is assumed that players can.

Concerning the solution part, the resulting payoffs to the players are determined according to certain solution concepts. The objectives of a solution theory can be different and hence it is not surprising that several distinct solution concepts have been developed on the same modelling theory. The usual distinction between noncooperative and cooperative theory is mainly methodological: whereas noncooperative game theory is strategy-oriented, in cooperative game theory the emphasis is on the feasibility of payoff vectors.

This monograph is devoted to cooperative game theory and in particular transferable utility games in characteristic function form, or TU-games for short. In what follows, Section 1.2 is an introduction of notations we use throughout this monograph. Section 1.3 provides a brief description of linearity and a basic idea of the algebraic approach to cooperative game theory. Finally, an overview of the monograph is given.

1.2 Cooperative games and solution concepts

In this section we first introduce the notation we use throughout this monograph. Next we formally introduce TU-games, as well as solution concepts.

1.2.1 Games and examples

In many economic and other social activities, participants achieve an agreement and form a coalition with the aim of maximizing the rewards or minimizing the costs which they can ensure themselves. Let us look at a typical example in economics.

Example 1.2.1 (Oil market game [84]). *Country 1 has oil which it can use to run its transport system at a profit of a dollars per barrel. Country 2 wants to buy the oil to use in its manufacturing industry, where it gives a profit of b dollars per barrel, while Country 3 wants it for food manufacturing where the profit is c dollars per barrel, $a < b \leq c$.*

We model this situation as a game with cooperation between three countries. Assume that Countries 1, 2, and 3 can cooperate freely for purchasing the maximal profit v that they can ensure no matter what the other(s) do. Then we have the following:

$v(\{1\}) = a$, because if 2 and 3 form a coalition against 1, they cannot force the seller to sell the oil to them, so it is worth a dollars to the seller;

$v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, because any coalition of buyers can not make any profits without oil;

$v(\{1,2\}) = b$, because 1 and 2 can use the oil at a profit of b dollars per barrel (1 sells it to 2), and so 3 would have to pay at least b dollars to get it;

$v(\{1,3\}) = v(\{1,2,3\}) = c$, since 1 and 3 can use 1's oil at a profit of c dollars per barrel. □

Formally, a *cooperative game* with transferable utility (TU-game) is a pair $\langle N, v \rangle$, where N is a nonempty, finite set called *player set* and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set 2^N of N, satisfying $v(\emptyset) = 0$. An element of N (notation: $i \in N$) and a subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) are called a *player* and *coalition* respectively. The associated real number $v(S)$ is called the *worth* of coalition S. The size of coalition S is denoted by s. Particularly, n denotes the size of the player set N. We denote by $\mathcal{G}^N$ the set of all these TU-games with player set N and by $\Omega = 2^N \setminus \emptyset$ the set of all coalitions. Throughout this monograph we speak of a cooperative game or simply a game when we refer to a transferable utility game. It is tacitly assumed that the worth of the empty set is zero whenever we refer to any game.

Given two games $\langle N, v \rangle, \langle N, w \rangle$ and scalar $\alpha \in \mathbb{R}$, with the usual operations of addition $\langle N, v + w \rangle$, defined by $(v + w)(S) = v(S) + w(S)$, and scalar multiplication $\langle N, \alpha v \rangle$, defined by $(\alpha v)(S) = \alpha v(S)$ for all $S \in \Omega$, the set $\mathcal{G}^N$ becomes a real vector space.

For a given game $\langle N, v \rangle$, its *dual game* $\langle N, v^* \rangle$ and *complementary game* $\langle N, \bar{v} \rangle$ are defined respectively as

$$v^*(S) = v(N) - v(N \setminus S), \quad \text{for all } S \in \Omega; \tag{1.2.1}$$

$$\bar{v}(S) = v(N \setminus S), \quad \text{for all } S \in \Omega. \tag{1.2.2}$$

A *subgame* $\langle S, v \rangle$ of $\langle N, v \rangle$ is the game with play set $S \subseteq N, S \neq \emptyset$ and with the same worth $v(T)$ as in the original game $\langle N, v \rangle$, for all $T \subseteq S$.

A game $\langle N, v \rangle$ is called *monotone* if $v(S) \leq v(T)$, for all $S, T \in \Omega$ with $S \subseteq T$, and it is called *superadditive* if

$$v(S) + v(T) \leq v(S \cup T), \quad \text{for each } S, T \in \Omega \text{ with } S \cap T = \emptyset.$$

Particularly, it is said to be *inessential* or *additive* if the equality $v(S) + v(T) = v(S \cup T)$ always holds, or equivalently, $v(S) = \sum_{i \in S} v(\{i\})$, for all $S \in \Omega$. Clearly, for any inessential game $\langle N, v \rangle$, its dual game agrees with the game itself, *i.e.*, $\langle N, v^* \rangle = \langle N, v \rangle$. We call this property the *self-duality* of inessential games.

A game $\langle N, v \rangle$ is said to be *constant-sum* if

$$v(S) + v(N \setminus S) = v(N), \quad \text{for all } S \in \Omega.$$

Obviously, inessential games are constant-sum, whereas constant-sum games are not necessarily superadditive. A game $\langle N, v \rangle$ is said to be *simple* if it is monotone and, for each $S \in \Omega$, either $v(S) = 0$ or $v(S) = 1$, and particularly, $v(N) = 1$.

Essentially, every coalition in a simple game is either winning (value 1) or losing (value 0), with nothing in between. As such, simple games are applicable to political sciences, as they include voting games in elections and legislatures. Let us turn to another example in social life and model it as a cooperative game.

Example 1.2.2 (Lilliput U.N. Security Council [84]). *Lilliput has a small version of the U.N. Security Council (the numbers get too large in the real situation) which has two permanent members, 1 and 2, who have the veto, plus three ordinary members, 3, 4 and 5. For a resolution to be passed, it requires at least three votes in favor and no vetoes.*

Suppose that a coalition is formed with the aim of passing the resolution. If the resolution is passed we give worth 1 to it, otherwise, worth 0. For the corresponding simple game, $v(S) = 1$ if and only if $\{1, 2\} \subseteq S$ and $s \geq 3$. The numerical form of this game is given by

$$v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = v(\{1, 2, 5\}) = v(\{1, 2, 3, 4\}) = v(\{1, 2, 3, 5\})$$

$$= v(\{1, 2, 4, 5\}) = v(\{1, 2, 3, 4, 5\}) = 1;$$

$$v(S) = 0, \quad \text{for all other } S \subseteq N.$$

Clearly, the Lilliput U.N. Security Council game is not constant-sum. $\square$

A game $\langle N, v \rangle$ is called *convex*, if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \quad \text{for all } S, T \in \Omega.$$

A game $\langle N, v \rangle$ is said to be *concave* if and only if $\langle N, -v \rangle$ is convex.

For a game $\langle N, v \rangle$, player $i \in N$, and coalition $S \in \Omega$, the *marginal contribution* of i to S in $\langle N, v \rangle$, denoted by $m_i^S(v)$, is given by

$$m_i^S(v) = \begin{cases} v(S) - v(S \setminus \{i\}), & \text{if } i \in S; \\ v(S \cup \{i\}) - v(S), & \text{if } i \notin S. \end{cases}$$

Convexity of a game $\langle N, v \rangle$ is equivalent to

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \quad \text{or} \quad m_i^S(v) \leq m_i^T(v),$$

for all $i \in N$ and $S, T \subseteq N \setminus \{i\}$ with $S \subseteq T$. Hence, if a game is convex, then any marginal contribution of a player to a coalition is at most the player's marginal contribution to a larger coalition. It is easy to check that both the oil market game and the Lilliput U.N. Security Council game are monotone as well as superadditive. Moveover, the Lilliput U.N. Security Council game is convex whereas the oil market game is not. We turn to a general game involving mathematical functions.

Example 1.2.3 (Square function game). *Let $(d_i)_{i=1}^n$ be a collection of nonnegative real numbers. By the square function $f(x) = x^2$ we define the game $\langle N, v \rangle$ as*

$$v(S) = \left(\sum_{j \in S} d_j \right)^2, \quad \text{for all } S \in \Omega.$$

Since $d_i \geq 0$ for every $i \in N$, the game verifies $v(S) \leq v(T)$ for all $S, T \in \Omega$ with $S \subseteq T$. The game $\langle N, v \rangle$ is monotone. Clearly, by $a^2 + b^2 \leq (a+b)^2$ for all $a, b \in \mathbb{R}_+$, we conclude that $\langle N, v \rangle$ is superadditive since $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in \Omega$ with $S \cap T = \emptyset$. For all $i \in N$ and $S, T \subseteq N \setminus \{i\}$ with $S \subseteq T$, we have the marginal contributions verifying the following:

$$m_i^S(v) = d_i^2 + 2d_i \sum_{j \in S} d_j \leq d_i^2 + 2d_i \sum_{j \in T} d_j = m_i^T(v).$$

Therefore, the game $\langle N, v \rangle$ is also convex. We can define a game $\langle N, v \rangle$ similarly by using any power function $f(x) = x^p$ with $p > 0$, such as an inessential game $\langle N, v \rangle$ with $p = 1$.   □

1.2.2   Solution concepts

If a coalition forms, then it may distribute its worth among its members in any way, and the central question in the framework of TU-games is: how will or should it do this? In studying this question, it is usually assumed that all the players who are participating in a TU-game will work together and form the grand coalition. Without loss of generality, if the formation of smaller coalitions applies, then the question of how to distribute their worths needs to be answered. On the other hand, the assumption avoids the equally important question of coalition formation and the interaction between coalition formation and payoff distribution.

The question of payoff distribution can be handled by proposing a solution, *i.e.*, a map or correspondence assigning a payoff distribution or a set of payoff distributions to every TU-game. Two related issues play a major role in the study of solutions. Firstly, a solution should be game-theoretically justified, which goal can be attained by axiomatic characterization. Secondly, the computation of a solution may be a difficult task. In particular because the number of coalitions grows exponentially with the number of players.

Formally, a *payoff vector* $x = (x_i)_{i \in N} \in \mathbb{R}^N$ of a TU-game $\langle N, v \rangle$ is an n-dimensional column vector allocating a payoff x_i to player $i \in N$. The so-called *payoff* x_i to player i in the game represents an assessment by i of his gains for participating in the game. A *solution mapping(function)* is a mapping (function) f that assigns to every game $\langle N, v \rangle$ a set of payoff vectors in $\mathbb{R}^N$, that is $f(v) \subseteq \mathbb{R}^N$.

For a payoff vector $x \in \mathbb{R}^N$ and coalition $S \in \Omega$, we denote by $x(S) = \sum_{i \in S} x_i$ the total payoff to the members of coalition S and next the *excess* of S in the game $\langle N, v \rangle$ with respect to x is given by

$$e(S, x) = v(S) - x(S).$$

Note that $e(S, x)$ can be interpreted as a measure of the dissatisfaction of coalition S if payoff vector x was suggested as final payoff: the greater $e(S, x)$, the more ill-treated S would feel.

In cooperative game theory there exist many solution concepts, each with its own reasonable rules for allocating. The reasonable rules of these solution concepts are mostly measured in terms of criteria or named properties. For a game $\langle N, v \rangle$, an allocation x is called *efficient* if it exactly distributes the profits of the coalition N in the game $\langle N, v \rangle$ among the players, *i.e.*, if

$$\sum_{i \in N} x_i = v(N), \quad \text{or} \quad x(N) = v(N).$$

The set of all efficient allocations in the game $\langle N, v \rangle$ is called the *pre-imputation set*, denoted by $I^*(v)$, *i.e.*,

$$I^*(v) = \{ x \in \mathbb{R}^N \mid x(N) = v(N) \}.$$

A widely accepted criterion for an allocation x in a game $\langle N, v \rangle$ is that every player should receive at least the amount he can obtain by operating on his own, *i.e.*,

$$x_i \geq v(\{i\}), \quad \text{for all } i \in N.$$

Allocations satisfying these inequalities are called *individually rational*. The (possibly empty) *imputation set $I(v)$* of a game $\langle N, v \rangle$ consists of all efficient and individually rational allocations:

$$I(v) = \{ x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}), \text{ for all } i \in N \}.$$

The criterion mentioned above could be strengthened by demanding that not only every player, but also every coalition $S \in \Omega$ should receive at least the worth it can obtain by operating on its own, *i.e.*,

$$x(S) \geq v(S), \quad \text{for all } S \in \Omega.$$

The *core* (Gillies [34]) consists of all efficient allocations satisfying the above inequalities:

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S), \text{ for all } S \in \Omega\}.$$

Let us look at the two games in our examples and find their cores.

Example 1.2.4 (Oil market game [84]). *Recall* $v(\{1\}) = a$, $v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, $v(\{1,2\}) = b$, $v(\{1,3\}) = v(\{1,2,3\}) = c$. *A payoff vector* $x = (x_1, x_2, x_3)$ *belongs to the core if and only if*

$$x_1 \geq a; \ x_2 \geq 0; \ x_3 \geq 0;$$
$$x_1 + x_2 \geq b; \ x_1 + x_3 \geq c; \ x_2 + x_3 \geq 0;$$
$$x_1 + x_2 + x_3 = c.$$

Thus, its core is given by the line segment

$$C(v) = \{(\lambda, 0, c - \lambda) \mid b \leq \lambda \leq c)\}.$$

We can give an interpretation to this result. It states that Countries 1 and 3 form a coalition and so, Country 1 will sell the oil to Country 3. Country 3 pays to the seller the amount of x dollars per barrel which must be at least b dollars, otherwise Country 1 would be better off selling the oil to Country 2, and no more than c dollars, so that Country 3 does not pay more than its worth to him. $\qquad \square$

Example 1.2.5 (Lilliput U.N. Security Council [84]). *Recall* $v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,2,5\}) = v(\{1,2,3,4\}) = v(\{1,2,3,5\}) = v(\{1,2,4,5\}) = v(\{1,2,3,4,5\}) = 1$, $v(S) = 0$, *for all other* $S \in \Omega$. *For any allocation* x *in the core, the following inequalities are required:*

$$x_i \geq 0, \quad i = 1, 2, 3, 4, 5;$$
$$x_1 + x_2 + x_3 \geq 1; \ x_1 + x_2 + x_4 \geq 1; \ x_1 + x_2 + x_5 \geq 1;$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

Thus, its core is given by the line segment

$$C(v) = \{(\lambda, 1 - \lambda, 0, 0, 0) \mid 0 \leq \lambda \leq 1)\}.$$

This suggests that all the power resides with the two permanent members with the veto. One way of looking at this is to suppose that in the future each country will be given a percentage of the votes and motions will be passed if more than 50% of the votes are cast for it. However, the percentage of votes given to each country has to be decided under the present voting system. Which distribution of the votes will get passed? The core suggests that only those which give all the votes to 1 and 2 will get through. This seems highly improbable at first, but on second thought we can see how this will happen. Suppose, initially, that 1,2 and 3 decide on a third of the votes each, then 1 and 2 can go to 4, who has no votes at present and offer him $\frac{1}{6}$ of the votes say, if the rest are shared between them, 4 would obviously agree to this, but then 1 and 2 could go to 5 and offer him the same deal for $\frac{1}{12}$ of the votes. He would also accept that, as would 3 for $\frac{1}{24}$ of the votes, and 4 for $\frac{1}{48}$ of the votes the second time around. Continuing this bargaining sequence, 1 and 2 could eventually guarantee themselves as much of the votes as they wanted. □

The core can be viewed as a solution concept in a sense that it assigns to every game a (possibly empty) set of reasonable allocations. We would like to say the core is a solution set of the linear system of one equality and several inequalities. Core allocations have the property that no coalition can improve its situation by splitting off from the grand coalition N. Hence, if core allocations exist for a game $\langle N, v \rangle$, then it is very likely that the grand coalition N forms, and that the profit $v(N)$ will be allocated according to some vector in the core.

Other well-known solution concepts are the *bargaining set* (Aumann and Maschler [2]), the *prekernel* (Maschler, Peleg and Shapley [62]) and the *kernel* (Davis and Maschler [14]). The fact that these solution concepts do not assign a single allocation to a game, but a possibly empty set of allocations, can be regarded as a disadvantage. Therefore, there is also some interest in solution concepts which assign to every game exactly one allocation. Such single-valued solution concepts are called *values*. Among these, the *Shapley value* (Shapley [75]), the *prenucleolus* (Sobolev [77]), the *nucleolus* (Schmeidler [74]) and the *τ-value* (Tijs [88], Driessen [25, 26]) are the best known.

Let $\theta(e(x))$ be the vector of all excesses $e(S, x)$, $S \in \Omega$, with respect to x, arranged in non-increasing order and let $\preceq_{lex}$ be the lexicographic order.

The *prenucleolus* of a game $\langle N, v \rangle$ is the unique pre-imputation $\nu^* \in I^*(v)$ satisfying

$$\theta(e(\nu^*)) \preceq_{lex} \theta(e(x)), \quad \text{for all } x \in I^*(v).$$

Similarly, for games with non-empty imputation set, the *nucleolus* of the game is the unique imputation $\nu \in I(v)$ satisfying

$$\theta(e(\nu)) \preceq_{lex} \theta(e(x)), \quad \text{for all } x \in I(v).$$

An advantage of the prenucleolus and the nucleolus is that both of them always lie in the core of the game if the core is non-empty. Existence and uniqueness of the prenucleous are proved in Sobolev [78], and of the nucleolus in Schmeidler [74]. In both cases the selection is justified by a principle of fairness and stability: to minimize the maximal complaint that a coalition might raise against a proposed payoff.

For a game $\langle N, v \rangle$, the *utopia vector* $\mu(v) \in \mathbb{R}^N$ is defined by

$$\mu_i(v) = v(N) - v(N \setminus \{i\}), \quad \text{for all } i \in N, \tag{1.2.3}$$

and the *minimal right vector* $a(v) \in \mathbb{R}^N$ by

$$a_i(v) = \max_{S \ni i} \left[v(S) - \sum_{j \in S \setminus \{i\}} \mu_j(v) \right], \quad \text{for all } i \in N.$$

A game $\langle N, v \rangle$ is called *quasi-balanced* if $a_i(v) \leq \mu_i(v)$, for all $i \in N$ and $\sum_{i \in N} a_i(v) \leq v(N) \leq \sum_{i \in N} \mu_i(v)$. For a quasi-balanced game $\langle N, v \rangle$, the *τ-value* $\tau(v)$ (Tijs [88]) is defined as the linear combination of the utopia vector and the minimal right vector that is efficient, *i.e.*,

$$\tau(v) = \lambda \mu(v) + (1 - \lambda) a(v),$$

with $\lambda \in [0, 1]$ such that $\sum_{i \in N} \tau_i(v) = v(N)$.

The utopia vector $\mu(v) \in \mathbb{R}^N$ is also called *separable cost vector* by Moulin [63] who introduced the *equal allocation of nonseparable cost value* (*EANSC-value*) as a single-valued solution that assigns to the game $\langle N, v \rangle$ the vector $EANSC(v) \in \mathbb{R}^N$ given by

$$EANSC_i(v) = \mu_i(v) + \frac{1}{n} \left[v(N) - \sum_{j \in N} \mu_j(v) \right], \quad \text{for all } i \in N. \tag{1.2.4}$$

Obviously, the *EANSC*-value is a pre-imputation, but not necessarily an imputation.

1.3 Linearity in cooperative game theory

In cooperative game theory, linearity is often studied as a subject and used as a tool in both modelling part and solution part. In this section we review some essential examples with respect to linearity, and afterwards the idea of the algebraic approach to the framework of cooperative games comes forward intuitively.

1.3.1 Linearity in the modelling part

We first define a *lexicographic order* for the set Ω of coalitions. In this order, for two coalitions $S = \{i_1, i_2, \cdots, i_s\}$ and $T = \{j_1, j_2, \cdots, j_t\}$ with $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$, S *precedes* T if and only if the sizes of these two coalitions verify either $s < t$, or $s = t$ and for some k, $1 \leq k < s$, it holds that $i_l = j_l$, for all $1 \leq l \leq k - 1$ and $i_k < j_k$.

In this monograph a game $\langle N, v \rangle$ is always presented as the column vector $\vec{v}$ of worths $v(S)$ of all lexicographically ordered coalitions $S \in \Omega$, *i.e.*,

$$\vec{v} = (v(S))_{S \in \Omega}.$$

For example, a 3-person game $\langle N, v \rangle$ will be presented as

$$\vec{v} = \big(v(\{1\}), v(\{2\}), v(\{3\}), v(\{1,2\}), v(\{1,3\}), v(\{2,3\}), v(\{1,2,3\})\big)'.$$

If no confusion arises, we write v instead of $\vec{v}$. In a sense, the set $\mathcal{G}^N$ of all n-person games with player set N is isomorphic to the vector space $\mathbb{R}^{2^n - 1}$, for always $v(\emptyset) = 0$. Corresponding to the standard basis for $\mathbb{R}^{2^n - 1}$, we introduce the basis for the game space $\mathcal{G}^N$ called unity games. With every coalition $S \in \Omega$, there is associated its *unity game* $\langle N, e_S \rangle$ defined by

$$e_S(T) = \begin{cases} 1, & \text{if } S = T; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.1}$$

Clearly, any game $\langle N, v \rangle$ can be represented as a linear combination of the unity games $\langle N, e_S \rangle$, $S \in \Omega$, such that

$$v = \sum_{S \in \Omega} v(S) \cdot e_S. \tag{1.3.2}$$

Shapley [75] introduced another well-known basis for the game space $\mathcal{G}^N$ called unanimity games. With every coalition $S \in \Omega$, there is associated its *unanimity game* $\langle N, u_S \rangle$ defined by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.3}$$

It is well-known that that any game $\langle N, v \rangle$ can be represented as a linear combination of the unanimity games $\langle N, u_S \rangle$, $S \in \Omega$, such that

$$v = \sum_{S \in \Omega} \Delta^v(S) \cdot u_S,$$

where $\Delta^v(S)$ is the so-called dividend with respect to the coalition S, $S \in \Omega$. The *dividends* $\Delta^v(S)$, $S \in \Omega$, of a game $\langle N, v \rangle$, as defined by Harsanyi [40,41], are of the form

$$\Delta^v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T), \quad \text{for all } S \in \Omega. \tag{1.3.4}$$

Observe that the characteristic functions of unanimity games are linearly independent, *i.e.*, the unanimity games form a basis for the game space $\mathcal{G}^N$. Moreover, the unanimity games are convex.

The dual games $\langle N, u_S^* \rangle$ of the unanimity games $\langle N, u_S \rangle$, $S \in \Omega$, form another basis for the game space $\mathcal{G}^N$. For any $S \in \Omega$, the dual game $\langle N, u_S^* \rangle$ is given by

$$u_S^*(T) = \begin{cases} 1, & \text{if } S \cap T \neq \emptyset; \\ 0, & \text{if } S \cap T = \emptyset. \end{cases} \tag{1.3.5}$$

We give the following interpretation of this game. The coalition S can be thought of as the set of all directors. Every coalition having at least one director will win the game with payoff 1. A coalition without any director will

lose the game and get payoff 0. The dual of a convex game is generally known and can easily be shown to be concave. Therefore, these dual games $\langle N, u_S^* \rangle$, $S \in \Omega$, are concave.

Recently, Driessen, Khmelnitskaya and Sales [29] studied another basis for the game space $\mathcal{G}^N$ called complementary unanimity games. With every coalition $S \in \Omega$, there is associated its *complementary unanimity game* $\langle N, \bar{u}_S \rangle$ defined by

$$\bar{u}_S(T) = \begin{cases} 1, & \text{if } S \cap T = \emptyset; \\ 0, & \text{if } S \cap T \neq \emptyset. \end{cases} \tag{1.3.6}$$

Obviously, $\bar{u}_S(T) = u_S(N \setminus T)$ for all $T \in \Omega$. For this reason, we call these games complementary unanimity games. Since $\bar{u}_N \equiv \vec{0}$, the game $\bar{u}_N$ is out of interest and so, it is replaced by another game $\bar{u}_\emptyset$ with reference to the empty set, defined by

$$\bar{u}_\emptyset(T) = 1, \quad \text{for all } T \in \Omega. \tag{1.3.7}$$

Therefore, when we mention the basis of complementary unanimity games, we refer to the collection $\{\langle N, \bar{u}_S \rangle \mid S \subsetneqq N\}$. Particularly, in the lexicographic order of this basis, the last one is indexed by the empty set $\emptyset$ instead of the grand coalition N (see Remark 2.1.5).

These complementary unanimity games can be interpreted as follows. The coalition S is considered as the set of all enemies. Any member(s) of S will cause a payoff 0. Every coalition without enemies will win the game and get payoff 1. Driessen *et al.* [29] showed that any game $\langle N, v \rangle$ can be represented as a linear form

$$v = \sum_{\substack{S \subsetneqq N}} \bar{\Delta}^v(S) \cdot \bar{u}_S, \tag{1.3.8}$$

where $\bar{\Delta}^v(S)$, $S \subsetneqq N$, are the so-called *complementary dividends* of the game $\langle N, v \rangle$, given by

$$\bar{\Delta}^v(S) = \sum_{N \setminus T \subseteq S} (-1)^{s-(n-t)} v(T). \tag{1.3.9}$$

Between any pair of bases of the game space $\mathcal{G}^N$, there is a nonsingular linear transformation operator. For more details about the relationships between these bases mentioned above, we refer to Chapters 2 and 6.

1.3.2   Linearity in the solution part

A well-established approach to the solution part, initiated by Shapley [75] with the introduction of the Shapley value, is to pose a list of desirable properties that fully characterize the solution. Let Φ be a value on $\mathcal{G}^N$. Firstly, let us review several essential properties treated in former axiomatizations.

- *Efficiency*: $\sum\limits_{i \in N} \Phi_i(v) = v(N)$, for all games $\langle N, v \rangle$;

- *Individual rationality*: $\Phi_i(v) \geq v(\{i\})$, for all games $\langle N, v \rangle$, and all $i \in N$;

- *Additivity*: $\Phi(v + w) = \Phi(v) + \Phi(w)$, for all games $\langle N, v \rangle$, $\langle N, w \rangle$;

- *Linearity*: $\Phi(\alpha \cdot v + \beta \cdot w) = \alpha \cdot \Phi(v) + \beta \cdot \Phi(w)$, for all games $\langle N, v \rangle$, $\langle N, w \rangle$, and all $\alpha, \beta \in \mathbb{R}$;

- *Symmetry* (*anonymity, equal treatment property*): $\Phi_{\pi(i)}(\pi v) = \Phi_i(v)$, for all games $\langle N, v \rangle$, all $i \in N$, and every permutation π on N. Here the game $\langle N, \pi v \rangle$ is given by $(\pi v)(S) = v(\pi^{-1}(S))$ for all $S \in \Omega$;

- *Null player property*: $\Phi_i(v) = 0$, for all games $\langle N, v \rangle$ and any null player $i \in N$. Player i is a *null player* in the game $\langle N, v \rangle$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$;

- *Dummy player property*: $\Phi_i(v) = v(\{i\})$, for all games $\langle N, v \rangle$ and any dummy player $i \in N$. Player i is a *dummy player* in the game $\langle N, v \rangle$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus \{i\}$;

- *Inessential game property*: $\Phi_i(v) = v(\{i\})$, for all inessential games $\langle N, v \rangle$ and all $i \in N$;

- *Monotonicity*: $\Phi_i(v) \geq 0$, for all monotonic games $\langle N, v \rangle$, and all $i \in N$;

- *Self-duality*: $\Phi_i(v) = \Phi_i(v^*)$, for all games $\langle N, v \rangle$ and its dual $\langle N, v^* \rangle$, and all $i \in N$;

- *Desirability* (*marginal contribution monotonicity*): $\Phi_i(v) \geq \Phi_j(v)$, if player i is more desirable than player j in any game $\langle N, v \rangle$. Player i is called more desirable than player j in the game $\langle N, v \rangle$ if $m_i^S(v) \geq m_j^S(v)$, i.e., $v(S \cup \{i\}) \geq v(S \cup \{j\})$, for all $S \subseteq N \setminus \{i, j\}$;

- *Substitution property*: $\Phi_i(v) = \Phi_j(v)$, for substitutes i and j in any game $\langle N, v \rangle$. Players i and j are called *substitutes* if both of them are more desirable, or equivalently, the equality for their marginal contributions $m_i^S(v) = m_j^S(v)$, *i.e.*, $v(S \cup \{i\}) = v(S \cup \{j\}))$, for all $S \subseteq N \setminus \{i, j\}$;

- *Covariance* (*strategic equivalence*): $\Phi(\alpha v + \beta) = \alpha \Phi(v) + \beta$, for all games $\langle N, v \rangle$, and all $\alpha \in (0, \infty)$, $\beta \in \mathbb{R}^N$. Here the game $\langle N, \alpha v + \beta \rangle$ is given by $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{j \in S} \beta_j$, for all $S \in \Omega$.

- *Continuity*: if for every convergent sequence of games $\{\langle N, v_k \rangle\}_{k=0}^{\infty}$, say the limit of which is the game $\langle N, \bar{v} \rangle$, the corresponding sequence of values $\{\Phi(v_k)\}_{k=0}^{\infty}$ converges to the value $\Phi(\bar{v})$.

For a value we remark that additivity can obviously be deduced from linearity, and they are equivalent for continuous values. Most of the solution concepts studied in this monograph are continuous. The dummy player property implies the null player property for a null player to be a dummy player with $v(\{i\}) = 0$, and implies the inessential game property for each player to be a dummy player in an inessential game. Symmetry implies the substitution property and, also, the desirability implies the substitution property. Linearity together with the inessential game property imply covariance. And monotonicity together with the dummy player property imply the desirability by choosing player j as a dummy player.

Stimulated by the fact that the game space $\mathcal{G}^N$ spans a vector space, linear values on $\mathcal{G}^N$ were well-studied and became the most important class of solution concepts in cooperative game theory. In the following we briefly go over several linear solution concepts involved in this monograph, of which the Shapley value is the most important representative.

The Shapley value

The Shapley value is the most well-known single-valued solution concept (see The Shapley value: Essays in honor of Lloyd S. Shapley [71]), which was introduced and characterized by Shapley [75] with the aid of four properties. Without going into details, we recall the following formula for the *Shapley*

value $Sh(v)$ of a game $\langle N, v \rangle$:

$$Sh_i(v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \Big[v(S) - v(S \setminus \{i\}) \Big], \quad \text{for all } i \in N. \quad (1.3.10)$$

For all $i \in N$, we have $\sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} = 1$. Thus, (1.3.10) deals with a probability distribution over the collection of coalitions containing player i. Note that this distribution arises from the belief that the coalition, to which player i belongs, is equally likely to be of any size s, $1 \leq s \leq n$, and that all such coalitions of size t are equally likely. If, for each $S \in \Omega$ with $S \ni i$, the coefficient $\frac{(s-1)!(n-s)!}{n!}$ is seen as the probability that i belongs to the coalition S and the marginal contribution $v(S) - v(S \setminus \{i\})$ is paid to i for its membership of S, then the Shapley value $Sh_i(v)$ for player i, as given above, is simply the expected payoff to player i in the game $\langle N, v \rangle$. For that very reason, Weber [99] called the Shapley value a probabilistic value. The Shapley value of the square function game of Example 1.2.3 is calculated in the following and turns out to be proportional to the nonnegative real numbers $(d_i)_{i \in N}$.

Example 1.3.1 (Square function game). *For any player $i \in N$, we have*

$$
\begin{aligned}
Sh_i(v) &= \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \Big[v(S) - v(S \setminus \{i\}) \Big] \\
&= \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \Big[d_i^2 + 2d_i \sum_{j \in S \setminus \{i\}} d_j \Big] \\
&= d_i^2 \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} + 2d_i \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \sum_{j \in S \setminus \{i\}} d_j \\
&= d_i^2 + 2d_i \sum_{j \in N \setminus \{i\}} d_j \sum_{\substack{S \ni i \\ S \ni j}} \frac{(s-1)!(n-s)!}{n!} \\
&= d_i^2 + 2d_i \sum_{j \in N \setminus \{i\}} d_j \sum_{s=2}^{n} \frac{(s-1)!(n-s)!}{n!} \binom{n-2}{s-2} \\
&= d_i^2 + 2d_i \sum_{j \in N \setminus \{i\}} \frac{d_j}{2} = d_i \sum_{j \in N} d_j. \qquad \qquad \square
\end{aligned}
$$

The Shapley value possesses all of these properties listed except for individual rationality. However, the Shapley value of a superadditive game is an

imputation, which may fall outside the core of the game. The Shapley value is also a striking example of the power of the axiomatic approach [13,31,33,37,75]. The eldest axiomatization of the Shapley value is stated by Shapley himself as follows.

Theorem 1.3.1 (Shapley [75]). *The Shapley value is the unique value on* $\mathcal{G}^N$ *satisfying efficiency, symmetry, linearity, and dummy player property.*

For any unanimity game $\langle N, u_S \rangle$, $S \in \Omega$, the Shapley value satisfies $Sh_i(u_S) = 0$ for all $i \notin S$ by the dummy player property and next, $Sh_i(u_S) = \frac{1}{s}$ for all $i \in S$ by efficiency and symmetry. This technique together with linearity, were the tools used by Shapley to complete the above axiomatization. In Chapters 2 and 3, we will pay attention to other axiomatizations of the Shapley value.

Various researchers presented a number of solution concepts in cooperative game theory, which are variations of the Shapley value, such as the Banzhaf value [8,55], semivalues [32], the least square values [73], the weighted Shapley values [1,49,66] and the weighted Banzhaf values [67], the Weber set [99], *etc.*

The family of semivalues

The concept of *semivalue* $SE(v)$ of a game $\langle N, v \rangle$ was introduced by Dubey, Neyman, and Weber [32], and is of the form

$$SE_i(v) = \sum_{S \ni i} p_s^n \Big[v(S) - v(S \setminus \{i\}) \Big], \quad \text{for all } i \in N, \tag{1.3.11}$$

where $p^n = (p_s^n)_{s=1}^n$ is a collection of nonnegative real numbers satisfying the *normalization condition*

$$\sum_{s=1}^n \binom{n-1}{s-1} p_s^n = 1. \tag{1.3.12}$$

That is, p^n is a probability distribution over coalitions containing player i, which assigns the same probability to coalitions of the same size. Thus, a semivalue allocates to each player the expected marginal contribution according to the probability distribution p^n. Clearly, for

$$p_s^n = \frac{(s-1)!(n-s)!}{n!}, \quad s = 1, 2, \cdots, n,$$

we get a particular semivalue, precisely the Shapley value. The *Banzhaf value* [8] is one of semivalues with

$$p_s^n = \frac{1}{2^{n-1}}, \quad s = 1, 2, \cdots, n.$$

It is a uniform probability distribution. The utopia vector $\mu(v)$ of (1.2.3) is also a semivalue with $p_n^n = 1$ and $p_s^n = 0$ for all $s = 1, 2, \cdots, n-1$. It is easy to check that the normalization condition is satisfied in all of these three cases, and it is obvious that there are many more semivalues. Dubey, Neyman, and Weber [32] proved that

Theorem 1.3.2 (Dubey *et al.* [32]). *Any value Φ on $\mathcal{G}^N$ possesses linearity, symmetry, monotonicity and inessential game property if and only if Φ is a semivalue.*

The family of semivalues possesses all of these properties listed except for efficiency, individual rationality and self-duality. In fact the Shapley value is the unique efficient semivalue, and this is a crucial requirement if one is looking for a solution that can be accepted by all the players. This motivated Ruiz *et al.* [73] to consider the additive efficient normalization for semivalues (see Chapter 5).

The family of least square values

For the prenucleolus of a game, since the sum of all the excesses is constant over the pre-imputation set, decreasing the highest excess entails increasing the other excesses altogether. But perhaps this may cause decreasing even further the excess of some coalitions. In order to avoid this problem and looking for an allocation in which all the excesses are similar, according to an egalitarian philosophy, Ruiz, Valenciano, and Zarzuelo [72] defined the least square prenucleolus. Instead of minimizing it according to the lexicographic order, this solution tries to select the efficient payoff vector for which the resulting excesses are closest to the average excess under the least square criterion. Formally, the *least square prenucleolus*, denoted by $LS\nu^*$, is the solution of the following optimization problem for any game $\langle N, v \rangle$:

$$\text{Problem 1: Minimize} \sum_{S \in \Omega} \left[e(S, x) - \bar{e}(v, x) \right]^2 \quad \text{s.t.} \sum_{i \in N} x_i = v(N),$$

where $\bar{e}(v, x) = \frac{1}{2^n - 1} \sum_{S \in \Omega} e(S, x)$ is the average excess at x.

The formula of the least square prenucleolus $LS\nu^*(v)$ is

$$LS\nu_i^*(v) = \frac{v(N)}{n} + \frac{1}{2^n - 1} \Big[\sum_{\substack{S \subsetneq N \\ S \ni i}} (n - s) \cdot v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} s \cdot v(S) \Big], \quad \text{for all } i \in N.$$

The least square prenucleolus satisfies efficiency, linearity (additivity), symmetry (substitution property), inessential game property, self-duality, and covariance. Ruiz *et al.* [72] introduced the property of a value Φ on $\mathcal{G}^N$, called average marginal contribution monotonicity, as follows.

- *Average marginal contribution monotonicity*: $\Phi_i(v) \geq \Phi_j(v)$ for all games $\langle N, v \rangle$, and for any pair of players i and j with

$$\sum_{S \subseteq N \setminus \{i,j\}} m_i^S(v) \geq \sum_{S \subseteq N \setminus \{i,j\}} m_j^S(v).$$

Obviously, average marginal contribution monotonicity implies marginal contribution monotonicity (desirability). The least square prenucleolus is characterized by the following four axioms.

Theorem 1.3.3 (Ruiz *et al.* [72]). *The least square prenucleolus is the unique value on $\mathcal{G}^N$ which satisfies linearity, efficiency, the inessential game property and average marginal contribution monotonicity.*

Ruiz, Valenciano, and Zarzuelo [73] extended this value to the family of least square values by considering the same optimization problem with the assumption allowing different weights for different coalitions. Let $m^n = (m_s^n)_{s=1}^n$ be a collection of nonnegative coalitional weights only indexed by the size of coalitions. The optimal solution of the following optimization problem

$$\text{Problem 2: Minimize} \quad \sum_{S \in \Omega} m_s^n \big[e(S, x) - \bar{e}(v, x) \big]^2 \quad \text{s.t.} \quad \sum_{i \in N} x_i = v(N)$$

is called the *least square value* with respect to the collection m^n. Actually, the weight of the grand coalition m_n^n is irrelevant for the optimal solution, because for any efficient payoff vector x it is $e(N, x) = 0$. Therefore, there is

a corresponding least square value for any weight collection. We denote the least square value for a game $\langle N, v \rangle$ by $LS^m(v)$ and it is given by

$$LS_i^m(v) = \frac{v(N)}{n} + \frac{1}{n\sigma}\Big[\sum_{\substack{S \subseteq N \\ S \neq \\ S \ni i}} (n-s)m_s^n \cdot v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} sm_s^n \cdot v(S) \Big], \quad \text{for all } i \in N,$$

(1.3.13)

where $\sigma = \sum_{s=1}^{n-1} \binom{n-2}{s-1} m_s^n$. We call the set of these values the *family of least square values.* This family of values is characterized by five axioms inclusive of another type of monotonicity, named coalitional monotonicity.

- *Coalitional monotonicity*: a value Φ on $\mathcal{G}^N$ is coalitionally monotonic if for any pair of games $\langle N, v \rangle, \langle N, w \rangle$ such that $v(S) \geq w(S)$ for some $S \in \Omega$, and $v(T) = w(T)$ for all $T \in \Omega, T \neq S$, it holds that $\Phi_i(v) \geq \Phi_i(w)$ for all $i \in S$.

Theorem 1.3.4 (Ruiz *et al.* [73]). *A value Φ on $\mathcal{G}^N$ possesses linearity, efficiency, symmetry, inessential game property and coalitional monotonicity if and only if Φ belongs to the least square family.*

So, the least square family is a subset of the class of linear, symmetric and efficient values. Driessen [30] presented an Equivalence Theorem with four equivalent statements for the class of linear, symmetric and efficient values. From the theorem, the formula (1.3.13) of a least square value LS^m can be rewritten as

$$LS_i^m(v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!}\Big[b_s^n \cdot v(S) - b_{s-1}^n \cdot v(S \setminus \{i\}) \Big], \quad \text{for all } i \in N.$$

(1.3.14)

where $b_s^n = s\binom{n-1}{s}\frac{m_s^n}{\sigma}$, $s = 1, 2, \cdots, n-1$ and $b_n^n = 1$. More precisely, any least square value is the Shapley value of a corresponding $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$, i.e., $LS^m(v) = Sh(\mathcal{B}v)$. The $\mathcal{B}$-scaled game (see [30]) is defined by $(\mathcal{B}v)(S) = b_s^n \cdot v(S)$ for all $S \in \Omega$. This work will be reviewed in Chapters 4 and 5.

The Shapley value satisfies all five properties in the above characterization system, so it belongs to the family of least square values. The collection of

weights with reference to the Shapley value is given by

$$m_s^n = \frac{(s-1)!(n-s-1)!}{(n-1)!}, \quad 1 \le s \le n-1.$$

In comparison to the coalitional monotonicity of the family of least square values, Young [103] defined another type of monotonicity named *strong monotonicity*, for characterizing the Shapley value. By this property, Young presented another axiomatization of the Shapley value without linearity.

- *Strong monotonicity*: a value Φ on $\mathcal{G}^N$ is strongly monotonic if for any pair of games $\langle N, v \rangle, \langle N, w \rangle$ and $i \in N$ such that $m_i^S(v) \ge m_i^S(w)$ for all $S \in \Omega$, it holds that $\Phi_i(v) \ge \Phi_i(w)$.

Theorem 1.3.5 (Young [103]). *The Shapley value is the unique value on $\mathcal{G}^N$ which satisfies efficiency, symmetry, and strong monotonicity.*

Generally speaking, Young's axiomatization of the Shapley value is not valid anymore on some subclass of games, such as simple games. Khmelnitskaya [50] showed that Young's axiomatization is still valid for the Shapley value defined on the class of nonnegative constant-sum games with nonzero worth of the grand coalition and on the entire class of constant-sum games as well.

The Weber set

We denote by Π^N the set of all permutations $\pi : N \to N$ on the player set N. Given a permutation $\pi \in \Pi^N$ assigning a rank number $\pi(i)$ to player i, we denote by π^i the set of all predecessors, that is all players with rank numbers smaller than or equal to the rank number of i in π, *i.e.*,

$$\pi^i = \{j \in N \mid \pi(j) \le \pi(i)\}.$$

Obviously, $i \in \pi^i$. Then the *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^N$ of a game $\langle N, v \rangle$ with respect to a permutation $\pi \in \Pi^N$ is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\}), \quad \text{for all } i \in N. \tag{1.3.15}$$

That is, player i receives its marginal contribution to the worth of the coalition consisting of all its predecessors in π. Let $\{r_\pi \mid \pi \in \Pi^N\}$ be a probability

distribution with $r_\pi \geq 0$ for all $\pi \in \Pi^N$ and $\sum_{\pi \in \Pi^N} r_\pi = 1$. A *random order value* $\varphi^r \in \mathbb{R}^N$ of a game $\langle N, v \rangle$ is defined by

$$\varphi_i^r(v) = \sum_{\pi \in \Pi^N} r_\pi m_i^\pi(v), \quad \text{for all } i \in N.$$

So, a random order value is a convex combination of the marginal contribution vectors, *i.e.*, $\varphi^r(v) = \sum_{\pi \in \Pi^N} r_\pi m^\pi(v)$. Every random order values possesses efficiency, linearity (additivity), inessential game property, monotonicity and covariance. Furthermore, it is characterized by four properties among them as follows.

Theorem 1.3.6 (Weber [99]). *A value Φ on $\mathcal{G}^N$ satisfies linearity, efficiency, the null player property and monotonicity if and only if Φ is a random order value.*

In particular, we remark that the Shapley value $Sh(v)$ equals the average of the marginal contribution vectors over all permutations, *i.e.*,

$$Sh_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi^N} m_i^\pi(v), \quad \text{for all } i \in N. \tag{1.3.16}$$

In view of this formula, a second probabilistic interpretation of the Shapley value is given as follows. The formation of the grand coalition N is seen as a sequential process in such a way that the players enter one by one and that the order in which the players are to join is determined by chance, with all $n!$ orderings on N being equally likely. If player i is paid his marginal contribution $m_i^\pi(v)$ for joining his predecessors with respect to a certain ordering on N, the expected payoff to player i in the game $\langle N, v \rangle$ is given by (1.3.16).

The *Weber mapping*, introduced by Weber [99], is the solution mapping that assigns to each game $\langle N, v \rangle$ the *Weber set $W(v)$* of payoff vectors, given by the convex hull of all marginal contribution vectors, *i.e.*,

$$W(v) = \text{Conv}\{m^\pi(v) \mid \pi \in \Pi^N\}.$$

Therefore, the Weber set is the collection of all random order values. Contrary to the core, the Weber set is always nonempty. It contains the core as a subset, as shown in the work of Weber [99]. A short proof, based on the Separation

Theorem, can be found in the work of Derks [15]. The core and the Weber set coincide if and only if the game is convex (see Shapley [76] and Ichiishi [48]).

From the review of linearity in both modelling and solution parts, cooperative games as well as linear values can be represented algebraically by vectors and matrices respectively. Properties of cooperative games and linear values can be studied in terms of these vectors and matrices. The algebraic representation and the matrix analysis to cooperative game theory come forward as natural and powerful. Some initial ideas related to the algebraic approach appeared in the literature (see Weber [99], Dragan [20]). Grabisch [36] presented the way to use matrices instead of operators to study a set of linear nonsingular functions to cooperative game theory. Kleinberg and Weiss [52] constructed a direct-sum decomposition of the null space of the Shapley value into invariant subspaces by using the representation theory of symmetric groups, and derived a characterization of a very general type of values, of which the Shapley value is one particular example. Recently, Hernandez-Lamoneda, Juarez and Sanchez-Sanchez advertised a natural representation theory in [43], by computing a direct sum decomposition for the game space $\mathcal{G}^N$ under the action of the symmetric group S_n. Following this scheme, all linear, symmetric solutions may be written as a sum of trivial maps, and well known results as well as new theorems and characterizations of certain classes of linear symmetric solutions are derived. However, the algebraic representation and the matrix analysis have not been used systematically. It is still a neglected technique in cooperative game theory.

1.4   Overview

In this section we give an overview of our contributions to the study of cooperative games. This monograph is devoted to the matrix approach to cooperative game theory. It concentrates on linear transformations and linear values on the game space, particularly the Shapley value and its variations.

Chapters 2 to 5 deal with the class of linear, symmetric and efficient values, of which the Shapley value is the most important representative. Chapter 2 provides the foundation of this monograph. We present some general terminologies and notations for the topics of the monograph. Linear operators (*e.g.*, game transformations and linear values) on the game space are represented

algebraically in terminology of coalitional matrices. The Moebius transformation matrix associated with the basis of unanimity games, the dual matrix associated with the dual operator as well as the basis of dual unanimity games, the complementary Moebius transformation matrix associated with the basis of complementary unanimity games, and the Shapley standard matrix associated with the Shapley value are especially described.

In Chapters 3 and 4, we study the consistency with respect to different types of associated games for the class of linear, symmetric and efficient values, *i.e.*, the value keeps invariant when adapting any game into another game called associated game. Several types of games are introduced on the game space, which are linear transformations of an original game with special structure. Each value of class is characterized by three axioms: the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency (respectively, the $\mathcal{B}$-dual similar associated consistency). Particularly, the Shapley value is characterized by the inessential game property, continuity, and associated consistency (respectively, dual similar associated consistency). As a by-product we use neither linearity nor the efficiency axiom. By analyzing the null space of the Shapley standard matrix, the inverse problems are discussed for the Shapley value and generalized linear, symmetric and efficient values. Concerning the matrix approach to these axiomatizations, we use three tables to summarize the results.

Chapter 5 is dedicated to the consistency with respect to several types of reduced games for the class of linear, symmetric and efficient values *i.e.*, the value keeps invariant when adapting any game into another game called reduced game. Most of the contributions are related to $\mathcal{B}$-consistency and linear consistency. The additive efficient normalization of semivalues, are studied as significant representatives of this class.

For a game, its dividend vector is exactly the Moebius transformation of the worth vector. Chapter 6 presents a matrix approach, by the Moebius transformation and the complementary Moebius transformation, to the Harsanyi set and the Weber set with reference to the dividends and the complementary dividends. The Harsanyi set is defined by the collection of dividend sharing systems, modelled as matrices, which distributes, for any coalition, its dividend among its members. The Moebius transformation translates any dividend sharing matrix into another type of sharing matrix, which is associated

with the worth vector instead of the dividend vector. The structure of these matrices reflects the properties of linearity, efficiency, the null player property, and positivity, which are used to characterize Harsanyi payoff vectors (Derks, Haller, and Peters [16]). By the inverse of the Moebius transformation of all marginal vectors, which are the extreme points of the Weber set, we develop a shorter and intuitive characterizing procedure for the Weber set by the Harsanyi payoff vectors. From the interrelationship between the dividends and the complementary dividends, the Harsanyi set and the Weber set are also discussed in terms of the complementary Moebius transformation. The extreme points of the Harsanyi set and the Weber set are also described in terms of the carrier of an extreme point of a linear system.

Publications underlying this monograph

[1] Xu, G., and T.S.H. Driessen, (2006), *Matrix approach to the Shapley value and dual similar associated consistency*, Memorandum No. 1797, Department of Mathematics, University of Twente, The Netherlands. Revision submitted to Linear Algebra and its Applications. (Chapter 3)

[2] Xu, G., H. Sun, C. Hoede and T.S.H. Driessen, (2007), *A solution defined by fine vectors*, Memorandum No. 1833, Department of Applied Mathematics, University of Twente, The Netherlands.   (Chapter 6)

[3] Xu, G., and T.S.H. Driessen, (2007), *Matrix approach to consistency of the additive efficient normalization of semivalues*, Memorandum No. 1854, Department of Applied Mathematics, University of Twente, The Netherlands.   (Chapter 5)

[4] Xu, G., T.S.H. Driessen and H. Sun, (2008), *Matrix analysis for associated consistency in cooperative game theory*, Linear Algebra and its Applications 428, pp. 1571-1586.   (Chapters 3 and 4)

[5] Xu, G., and T.S.H. Driessen, (2008), *Moebius transformation and its complementary transformation in cooperative game theory*, Memorandum (in preparation).   (Chapters 2 and 6)

[6] Xu, G., and T.S.H. Driessen, (2008), *Matrix approach to the Weber set*, Memorandum (in preparation).   (Chapter 6)

Chapter 2

Matrix analysis for the Shapley value

In this chapter, we aim to introduce and develop the algebraic representation and the matrix approach in the framework of cooperative game theory. The notion of a coalitional matrix is defined and a linear transformation on the game space will be identified with a corresponding square-coalitional matrix. The four bases for the game space: unity games, unanimity games, dual unanimity games and complementary unanimity games, are interrelated by three square-coalitional matrices, which are associated with three types of linear operators: dual operator, Moebius operator and complementary Moebius operator. Furthermore, for any linear value, the payoff vector of any game is represented algebraically by the product of a column-coalitional representation matrix and the worth vector. Hence, the analysis of the structure of these representation matrices covers the study of the class of linear values. We achieve a matrix approach for the classic axiomatization of the Shapley value. And some properties are described for the Shapley standard matrix, which is the representation matrix of the Shapley value. Matrix analysis turns out to be a new and powerful technique for research in the field of cooperative game theory.

2.1   The coalitional matrix

We define a new type of matrix in order to apply algebraic representation theory and matrix theory to cooperative game theory.

Definition 2.1.1. *A matrix M is called row (respectively, column)-coalitional if its rows (respectively, columns) are indexed by all lexicographically ordered coalitions $S \in \Omega$. M is called square-coalitional if it is both row-coalitional and column-coalitional.*

For any game $\langle N, v \rangle$, the column vector $v = (v(S))_{S \in \Omega}$ of worths can be seen as a special row-coalitional matrix (vector). Similarly, we denote the dividend system and the complementary dividend system by the column vectors $\Delta^v = \left(\Delta^v(S) \right)_{S \in \Omega}$ and $\bar{\Delta}^v = \left(\bar{\Delta}^v(S) \right)_{S \in \Omega}$ respectively, which are indexed by all lexicographically ordered coalitions $S \in \Omega$. So, Δ^v and $\bar{\Delta}^v$ are also special row-coalitional matrices (vectors).

As mentioned in Subsection 1.3.1, for the reason that the game space $\mathcal{G}^N$ is isomorphic to the vector space $\mathbb{R}^{2^n - 1}$, linear transformation operators are widely used in cooperative game theory (see Grabisch, Marichal, and Roubens [35], and a survey by Grabish [36]). In view of (1.3.4) and (1.3.9), the dividend vectors Δ^v and $\bar{\Delta}^v$ are exactly linear transformations on the worth vector v, *i.e.*, the Moebius transformation and the complementary Moebius transformation. Another well-known example is the linear transformation of any original game $\langle N, v \rangle$ into its dual game $\langle N, v^* \rangle$. We recite and represent these three linear transformation operators by the corresponding square-coalitional matrices, which are well studied in this monograph.

Definition 2.1.2. *Given any game $\langle N, v \rangle$, its dividend vector Δ^v of (1.3.4) is represented by the Moebius transformation matrix M^Δ as*

$$\Delta^v = M^\Delta \cdot v, \tag{2.1.1}$$

where $M^\Delta = \left[M^\Delta \right]_{S,T \in \Omega}$ is square-coalitional defined by

$$\left[M^\Delta \right]_{S,T} = \begin{cases} (-1)^{s-t}, & \text{if } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.2}$$

Definition 2.1.3. *Given any game $\langle N, v \rangle$, its dual game $\langle N, v^* \rangle$ of (1.2.1) is represented by the dual matrix Q as*

$$v^* = Q \cdot v, \tag{2.1.3}$$

where $Q = [Q]_{S,T \in \Omega}$ *is square-coalitional defined by*

$$[Q]_{S,T} = \begin{cases} -1, & \text{if } T = N \setminus S \text{ and } S \neq N; \\ 1, & \text{if } T = N; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.4}$$

Definition 2.1.4. *Given any game* $\langle N, v \rangle$, *its complementary dividend vector* $\bar{\Delta}^v$ *of* (1.3.9) *is represented by the complementary Moebius transformation matrix* $M^{\bar{\Delta}}$ *as*

$$\bar{\Delta}^v = M^{\bar{\Delta}} \cdot v, \tag{2.1.5}$$

where $M^{\bar{\Delta}} = [M^{\bar{\Delta}}]_{S \subsetneq N, T \in \Omega}$ *is square-coalitional defined by*

$$[M^{\bar{\Delta}}]_{S,T} = \begin{cases} (-1)^{s-(n-t)}, & \text{if } N \setminus T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.6}$$

Remark 2.1.5. *The complementary set of* N *is* $\emptyset$, *which is out of the collection* Ω *of coalitions. In this framework, the lexicographically ordered coalitions are adapted such that the last one is always indexed by the empty set* $\emptyset$ *instead of the grand coalition* N. *The last row of the complementary Moebius transformation matrix* $M^{\bar{\Delta}}$ *is given by*

$$[M^{\bar{\Delta}}]_{\emptyset,T} = 0, \quad \text{for all } T \in \Omega, T \neq N, \quad \text{whereas } [M^{\bar{\Delta}}]_{\emptyset,N} = 1. \tag{2.1.7}$$

The following propositions are directly derived from the above definitions. Denote by $\mathbf{1}_\Omega \in \mathbb{R}^{2^n - 1}$ the column vector with all entries equal to one, and by I the identity matrix.

Proposition 2.1.6. *For the Moebius transformation matrix* M^{Δ}, *each column sum equals 0 except for the unitary sum of the last column indexed by the grand coalition* N, *i.e.,*

$$\mathbf{1}'_\Omega \cdot M^{\Delta} = (0, 0, \cdots, 0, 1).$$

Proof. For any column indexed by coalition $T \in \Omega$, we derive from (2.1.2)

and the Binomial Theorem that

$$\sum_{S\in\Omega}[M^\Delta]_{S,T} = \sum_{S\supseteq T}(-1)^{s-t} = \sum_{s=t}^{n}(-1)^{s-t}\binom{n-t}{s-t}$$

$$= \sum_{k=0}^{n-t}(-1)^k\binom{n-t}{k} = (1-1)^{n-t}$$

$$= \begin{cases} 1, & \text{if } t = n, \text{ i.e., } T = N; \\ 0, & \text{otherwise.} \end{cases}$$

That is to say, $\mathbf{1}'_\Omega \cdot M^\Delta = (0, 0, \cdots, 0, 1)$. $\qquad\square$

Proposition 2.1.7. *The dual matrix Q satisfies $Q^2 = I$. That is to say, for every game $\langle N, v\rangle$, the dual game $\langle N, (v^*)^*\rangle$ of its dual game $\langle N, v^*\rangle$ is the initial game $\langle N, v\rangle$.*

Proof by the matrix approach. Fix any column indexed by $T \in \Omega$. For a coalition $S \in \Omega$, $S \neq N$, we have

$$[Q^2]_{S,T} = \sum_{R\in\Omega}[Q]_{S,R}[Q]_{R,T}$$

$$= [Q]_{S,N\setminus S}[Q]_{N\setminus S,T} + [Q]_{S,N}[Q]_{N,T} = -[Q]_{N\setminus S,T} + [Q]_{N,T}$$

$$= \begin{cases} 1, & \text{if } T = S; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$[Q^2]_{N,T} = \sum_{R\in\Omega}[Q]_{N,R}[Q]_{R,T} = [Q]_{N,N}[Q]_{N,T} = [Q]_{N,T}$$

$$= \begin{cases} 1, & \text{if } T = N; \\ 0, & \text{otherwise.} \end{cases}$$

So $Q^2 = I$, and for any game $\langle N, v\rangle$, $(v^*)^* = Q \cdot v^* = Q \cdot Q \cdot v = v$. $\qquad\square$

Clearly, if we carry out a nonsingular linear transformation on any basis for a vector space, then another basis is derived. In this way, we can get some

appropriate bases for the game space $\mathcal{G}^N$. There are four different bases for G^N introduced in Subsection 1.3.1. In the following, the interrelationships between these four bases are presented with reference to the corresponding linear transformations.

Theorem 2.1.8. *The bases for the game space $\mathcal{G}^N$ satisfy the following:*

1. *The basis of dual unanimity games $\{\langle N, u_S^* \rangle \mid S \in \Omega\}$ is the dual transformation of the basis of unanimity games $\{\langle N, u_S \rangle \mid S \in \Omega\}$, i.e.,*

$$u_S^* = Q \cdot u_S, \quad \text{for all } S \in \Omega.$$

2. *The basis of unity games $\{\langle N, e_S \rangle \mid S \in \Omega\}$ is the Moebius transformation of the basis of unanimity games $\{\langle N, u_S \rangle \mid S \in \Omega\}$, i.e.,*

$$e_S = M^\Delta \cdot u_S, \quad \text{for all } S \in \Omega.$$

3. *The basis of unity games $\{\langle N, e_S \rangle \mid S \in \Omega\}$ is the complementary Moebius transformation of the basis of complementary unanimity games $\{\langle N, \bar{u}_S \rangle \mid S \subsetneq N\}$, i.e.,*

$$e_N = M^{\bar{\Delta}} \cdot \bar{u}_\emptyset, \quad \text{and} \quad e_S = M^{\bar{\Delta}} \cdot \bar{u}_S, \quad \text{for all } S \subsetneq N, S \neq \emptyset.$$

Proof. 1. It is derived directly from Definition 2.1.3.

2. For any unanimity game u_S, $S \in \Omega$, and any $T \in \Omega$, by (2.1.2), we have

$$\left[M^\Delta \cdot u_S \right](T) = \sum_{R \in \Omega} [M^\Delta]_{T,R} u_S(R) = \sum_{R \subseteq T} (-1)^{t-r} u_S(R)$$

$$= \sum_{\substack{R \subseteq T \\ R \supseteq S}} (-1)^{t-r} = \sum_{r=s}^{t} (-1)^{t-r} \binom{t-s}{r-s}$$

$$= (-1)^{t-s} \sum_{k=0}^{t-s} (-1)^k \binom{t-s}{k} = (-1)^{t-s} (1-1)^{t-s}$$

$$= \begin{cases} 1, & \text{if } t = s, \text{ i.e., } T = S \\ 0, & \text{otherwise} \end{cases}$$

$$= e_S(T).$$

Therefore, $e_S = M^\Delta \cdot u_S$, for all $S \in \Omega$.

3. For any complementary unanimity game $\bar{u}_S$, $S \subsetneq N$, by (2.1.6), we have

$$\left[M^{\bar{\Delta}} \cdot \bar{u}_S \right](N) = \sum_{R \in \Omega} [M^{\bar{\Delta}}]_{\emptyset, R} \bar{u}_S(R) = [M^{\bar{\Delta}}]_{\emptyset, N} \bar{u}_S(N) = \bar{u}_S(N)$$

$$= \begin{cases} 1, = e_N(N), & \text{if } S = \emptyset; \\ 0, = e_S(N), & \text{otherwise.} \end{cases}$$

If $S = \emptyset$, then $\bar{u}_\emptyset(R) = 1$ for all $R \in \Omega$ and so, for any coalition $T \in \Omega, T \neq N$,

$$\left[M^{\bar{\Delta}} \cdot \bar{u}_\emptyset \right](T) = \sum_{R \in \Omega} [M^{\bar{\Delta}}]_{T,R} \bar{u}_\emptyset(R) = \sum_{N \setminus R \subseteq T} (-1)^{t - (n-r)}$$

$$= \sum_{N \setminus T \subseteq R} (-1)^{t - (n-r)} = \sum_{r=n-t}^{n} (-1)^{t - (n-r)} \binom{t}{r - (n-t)}$$

$$= \sum_{k=0}^{t} (-1)^k \binom{t}{k} = (1-1)^t = 0$$

$$= e_N(T).$$

If $S \neq \emptyset$, for any coalition $T \in \Omega, T \neq N$, we have

$$\left[M^{\bar{\Delta}} \cdot \bar{u}_S \right](T) = \sum_{R \in \Omega} [M^{\bar{\Delta}}]_{T,R} \bar{u}_S(R) = \sum_{N \setminus R \subseteq T} (-1)^{t - (n-r)} \bar{u}_S(R)$$

$$= \sum_{\substack{N \setminus T \subseteq R \\ R \cap S = \emptyset}} (-1)^{t - (n-r)} = \sum_{\substack{N \setminus T \subseteq R \\ R \subseteq N \setminus S}} (-1)^{t - (n-r)}$$

$$= \sum_{r=n-t}^{n-s} (-1)^{t - (n-r)} \binom{t-s}{r - (n-t)} = \sum_{k=0}^{t-s} (-1)^k \binom{t-s}{k}$$

$$= (1-1)^{t-s} = \begin{cases} 1, & \text{if } t = s, i.e., T = S \\ 0, & \text{otherwise} \end{cases}$$

$$= e_S(T).$$

So, $e_N = M^{\bar{\Delta}} \cdot \bar{u}_\emptyset$, and $e_S = M^{\bar{\Delta}} \cdot \bar{u}_S$, for all $S \subsetneq N$, $S \neq \emptyset$. $\qquad \square$

2.2 Matrix approach to linear values

In this section we represent algebraically linear values in terminology of coalitional matrices. Next we apply matrix analysis to investigate characterizations of the class of linear values. First of all, by linear algebra we know that any linear value on game space can be represented uniquely by a corresponding column-coalitional matrix.

Theorem 2.2.1 (Linear algebra result [54]). *For any value Φ on $\mathcal{G}^N$, it is linear if and only if there exists a unique column-coalitional matrix $M^{\Phi} = \left[M^{\Phi}\right]_{i \in N, T \in \Omega}$ such that $\Phi(v) = M^{\Phi} \cdot v$, for all games $\langle N, v \rangle$.*

For the determination of the matrix M^{Φ}, recall, by (1.3.2), that any game $\langle N, v \rangle$ can be represented as $v = \sum_{S \in \Omega} v(S) \cdot e_S$. By the linearity of Φ, we have

$$\Phi(v) = \sum_{S \in \Omega} v(S) \cdot \Phi(e_S) = M^{\Phi} \cdot v,$$

where the entries of the column-coalitional matrix M^{Φ} are given by $\left[M^{\Phi}\right]_{i,S} = \Phi_i(e_S)$, for all $i \in N, S \in \Omega$.

Therefore, for any linear value, the payoff vector of any game is represented algebraically by the product of a column-coalitional matrix and the worth vector. We call this associated matrix the *representation matrix* of the linear value. In order to study the linear value, we may analyze the structure of this representation matrix. We start with linear values which possess some other essential properties listed in Section 1.3.2. Denote by $\mathbf{1}_N \in \mathbb{R}^N$ the n-dimensional column vector with all entries equal to one.

Proposition 2.2.2. *Let Φ be a linear value on $\mathcal{G}^N$. Then Φ is efficient if and only if each column sum of the representation matrix M^{Φ} equals 0 except for the unitary sum of the last column indexed by N, i.e.,*

$$\mathbf{1}_N' \cdot M^{\Phi} = (0, 0, \cdots, 0, 1).$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ and M^{Φ} be its representation matrix.

For any game $\langle N, v \rangle$ and any player $i \in N$, by Theorem 2.2.1, it follows that

$$\sum_{i \in N} \Phi_i(v) = \sum_{i \in N} \sum_{S \in \Omega} [M^\Phi]_{i,S} v(S) = \sum_{S \in \Omega} \sum_{i \in N} [M^\Phi]_{i,S} v(S)$$

$$= \sum_{\substack{S \in \Omega \\ S \neq N}} v(S) \sum_{i \in N} [M^\Phi]_{i,S} + v(N) \sum_{i \in N} [M^\Phi]_{i,N}.$$

Note that the game $\langle N, v \rangle$ is arbitrary and so, the worths $v(S)$, $S \in \Omega$, can be chosen arbitrarily. Thus, Φ is efficient, $i.e.$, $\sum\limits_{i \in N} \Phi_i(v) = v(N)$ for all games $\langle N, v \rangle$, if and only if

$$\sum_{i \in N} [M^\Phi]_{i,S} = 0, \text{ for all } S \in \Omega, S \neq N, \text{ and } \sum_{i \in N} [M^\Phi]_{i,N} = 1.$$

That is, $\mathbf{1}'_N \cdot M^\Phi = (0, 0, \cdots, 0, 1)$. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 2.2.3. *Let Φ be a linear value on $\mathcal{G}^N$. Then Φ possesses the null player property if and only if the representation matrix M^Φ satisfies the condition*

$$[M^\Phi]_{i,S} = -[M^\Phi]_{i,S \setminus \{i\}}, \quad \text{for all } i \in N, \text{ and all } S \in \Omega, S \ni i, S \neq \{i\}.$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ and M^Φ be its representation matrix.

"$\Leftarrow$": Suppose that M^Φ satisfies

$$[M^\Phi]_{i,S} = -[M^\Phi]_{i,S \setminus \{i\}}, \quad \text{for all } i \in N, \text{ and all } S \in \Omega, S \ni i, S \neq \{i\}.$$

For any game $\langle N, v \rangle$, by Theorem 2.2.1, we have

$$\Phi_i(v) = \sum_{S \in \Omega} [M^\Phi]_{i,S} v(S) = \sum_{S \ni i} [M^\Phi]_{i,S} v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} [M^\Phi]_{i,S} v(S)$$

$$= \sum_{\substack{S \ni i \\ S \neq \{i\}}} [M^\Phi]_{i,S} v(S) + [M^\Phi]_{i,\{i\}} v(\{i\}) + \sum_{\substack{S \ni i \\ S \neq \{i\}}} [M^\Phi]_{i,S \setminus \{i\}} v(S \setminus \{i\})$$

$$= \sum_{\substack{S \ni i \\ S \neq \{i\}}} [M^\Phi]_{i,S} \big[v(S) - v(S \setminus \{i\}) \big] + [M^\Phi]_{i,\{i\}} v(\{i\}).$$

For any null player i in the game $\langle N, v \rangle$, it holds that $v(\{i\}) = 0$ as well as $v(S) = v(S \setminus \{i\})$ for all $S \in \Omega$, $S \ni i, S \neq \{i\}$. Hence, $\Phi_i(v) = 0$, and so, Φ satisfies the null player property.

"$\Rightarrow$": For any $i \in N$, any $S \in \Omega$, $S \ni i$, $S \neq \{i\}$, consider the game $\langle N, v_{S_i} \rangle$ given by

$$v_{S_i}(S) = v_{S_i}(S \setminus \{i\}) = 1, \text{ and for any other coalition } T \in \Omega, \ v_{S_i}(T) = 0.$$

By Theorem 2.2.1, we obtain that

$$\Phi_i(v_{S_i}) = \sum_{R \in \Omega} [M^\Phi]_{i,R} v_{S_i}(R) = [M^\Phi]_{i,S} + [M^\Phi]_{i,S\setminus\{i\}}.$$

Obviously, i is a null player in the game $\langle N, v_{S_i} \rangle$. By the null player property, $\Phi_i(v_{S_i}) = 0$. Therefore, $[M^\Phi]_{i,S} = -[M^\Phi]_{i,S\setminus\{i\}}$. $\qquad\qquad\square$

Consider a weight system $m = (m_{i,S})_{i \in S}^{S \in \Omega}$ such that

$$\sum_{S \ni i} m_{i,S} = 1, \quad \text{for all } i \in N.$$

Weber introduced in [99] a value Φ on $\mathcal{G}^N$ as follows:

$$\Phi_i(v) = \sum_{S \ni i} m_{i,S} \big[v(S) - v(S \setminus \{i\}) \big], \quad \text{for all games } \langle N, v \rangle, \text{ and all } i \in N.$$

It is called a *Weber value* by Derks [18]. And Weber's characterization [99] of the class of Weber values is derived directly from Proposition 2.2.3.

Corollary 2.2.4 (Weber [99]). *A value Φ on $\mathcal{G}^N$ possesses linearity and the dummy player property if and only if it is a Weber value.*

Proof by the matrix approach. It is easy to check that every Weber value satisfies linearity and the dummy player property.

Let Φ be a linear value on $\mathcal{G}^N$ possessing the dummy player property and M^Φ be its representation matrix. Since for the value Φ, the dummy player property implies the null player property. By Proposition 2.2.3, for all $i \in N$, we have $[M^\Phi]_{i,S} = -[M^\Phi]_{i,S\setminus\{i\}}$ for all $S \in \Omega$, $S \ni i, S \neq \{i\}$. So, for any game $\langle N, v \rangle$, since $v(\emptyset) = 0$, we conclude that

$$\Phi_i(v) = \sum_{S \in \Omega} [M^\Phi]_{i,S} v(S) = \sum_{S \ni i} [M^\Phi]_{i,S} v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} [M^\Phi]_{i,S} v(S)$$

$$= \sum_{\substack{S \ni i \\ S \neq \{i\}}} [M^\Phi]_{i,S} v(S) + [M^\Phi]_{i,\{i\}} v(\{i\}) + \sum_{\substack{S \ni i \\ S \neq \{i\}}} [M^\Phi]_{i,S\setminus\{i\}} v(S \setminus \{i\})$$

$$= \sum_{S \ni i} [M^\Phi]_{i,S} \big[v(S) - v(S \setminus \{i\}) \big]$$

Let $\langle N, v \rangle$ be a game with the dummy player i. Then $v(S) - v(S \setminus \{i\}) = v(\{i\})$, for all $S \in \Omega, S \ni i$. By the dummy player property, it holds that $\Phi_i(v) = v(\{i\})$. So $\sum_{S \ni i} [M^\Phi]_{i,S} = 1$. Therefore, Φ is the Weber value with the weights $m_{i,S} = [M^\Phi]_{i,S}$, for all $S \in \Omega$, $S \ni i$. $\square$

Proposition 2.2.5. *Let Φ be a linear value on $\mathcal{G}^N$. Then Φ is symmetric if and only if the representation matrix M^Φ satisfies, for all players $i, j \in N$, and all coalitions $S, T \in \Omega$ with equal sizes $s = t$,*

$$[M^\Phi]_{i,S} = [M^\Phi]_{j,T}, \ \textit{when } i \in S, j \in T \textit{ or } i \notin S, j \notin T. \qquad (2.2.1)$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ and M^Φ be its representation matrix.

"$\Leftarrow$": Let $\pi \in \Pi^N$ be a permutation on N. Then for any $i \in N$ and any $T \in \Omega$, $\pi(i) \in \pi(T)$ if and only if $i \in T$, as well as $\pi(i) \notin \pi(T)$ if and only if $i \notin T$. So by (2.2.1), we have

$$[M^\Phi]_{\pi(i),\pi(T)} = [M^\Phi]_{i,T}.$$

Therefore, for all $i \in N$,

$$
\begin{aligned}
\Phi_{\pi(i)}(\pi v) &= \sum_{S \in \Omega} [M^\Phi]_{\pi(i),S}(\pi v)(S) \\
&= \sum_{S \ni \pi(i)} [M^\Phi]_{\pi(i),S}(\pi v)(S) + \sum_{\substack{S \not\ni \pi(i) \\ S \neq \emptyset}} [M^\Phi]_{\pi(i),S}(\pi v)(S) \\
&= \sum_{\pi^{-1}(S) \ni i} [M^\Phi]_{\pi(i),S} v(\pi^{-1}(S)) + \sum_{\substack{\pi^{-1}(S) \not\ni i \\ \pi^{-1}(S) \neq \emptyset}} [M^\Phi]_{\pi(i),S} v(\pi^{-1}(S)) \\
&= \sum_{T \ni i} [M^\Phi]_{\pi(i),\pi(T)} v(T) + \sum_{\substack{T \not\ni i \\ T \neq \emptyset}} [M^\Phi]_{\pi(i),\pi(T)} v(T) \\
&= \sum_{T \ni i} [M^\Phi]_{i,T} v(T) + \sum_{\substack{T \not\ni i \\ T \neq \emptyset}} [M^\Phi]_{i,T} v(T) \\
&= \Phi_i(v).
\end{aligned}
$$

The symmetry property for Φ holds.

"$\Rightarrow$": Suppose that Φ satisfies the symmetry property. For any players $i, j \in N, i \neq j$, since any permutation can be represented by a composition

of interchanges, firstly, we consider the interchange permutation π_{ij} which permutes only i to j and j to i. So for all $R \subseteq N \setminus \{i, j\}$,

$$(\pi_{ij}v)(R \cup \{j\}) = v(R \cup \{i\}), \quad (\pi_{ij}v)(R \cup \{i\}) = v(R \cup \{j\}); \quad (2.2.2)$$

and $\quad (\pi_{ij}v)(R) = v(R), \quad (\pi_{ij}v)(R \cup \{i, j\}) = v(R \cup \{i, j\}).$ $\qquad (2.2.3)$

For any game $\langle N, v \rangle$, we have

$$\Phi_{\pi_{ij}(i)}(\pi_{ij}v) = \sum_{S \ni \pi_{ij}(i)} [M^\Phi]_{\pi_{ij}(i),S}(\pi_{ij}v)(S) + \sum_{\substack{S \not\ni \pi_{ij}(i) \\ S \neq \emptyset}} [M^\Phi]_{\pi_{ij}(i),S}(\pi_{ij}v)(S)$$

$$= \sum_{S \ni j} [M^\Phi]_{j,S}(\pi_{ij}v)(S) + \sum_{\substack{S \not\ni j \\ S \neq \emptyset}} [M^\Phi]_{j,S}(\pi_{ij}v)(S), \qquad (2.2.4)$$

$$\Phi_i(v) = \sum_{S \ni i} [M^\Phi]_{i,S}v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} [M^\Phi]_{i,S}v(S) \qquad (2.2.5)$$

Taking $(2.2.5) - (2.2.4)$, by $(2.2.2)$, $(2.2.3)$, we have

$$\Phi_i(v) - \Phi_{\pi_{ij}(i)}(\pi_{ij}v)$$

$$= \sum_{\substack{S \ni i \\ S \not\ni j}} [M^\Phi]_{i,S}v(S) + \sum_{\substack{S \ni i \\ S \ni j}} [M^\Phi]_{i,S}v(S) + \sum_{\substack{S \not\ni i \\ S \ni j}} [M^\Phi]_{i,S}v(S) + \sum_{\substack{S \not\ni i, S \not\ni j \\ S \neq \emptyset}} [M^\Phi]_{i,S}v(S)$$

$$- \sum_{\substack{S \ni j \\ S \not\ni i}} [M^\Phi]_{j,S}(\pi_{ij}v)(S) - \sum_{\substack{S \ni j \\ S \ni i}} [M^\Phi]_{j,S}(\pi_{ij}v)(S)$$

$$- \sum_{\substack{S \not\ni j \\ S \ni i}} [M^\Phi]_{j,S}(\pi_{ij}v)(S) - \sum_{\substack{S \not\ni j, S \not\ni i \\ S \neq \emptyset}} [M^\Phi]_{j,S}(\pi_{ij}v)(S)$$

$$= \sum_{R \subseteq N \setminus \{i,j\}} [M^\Phi]_{i,R \cup \{i\}}v(R \cup \{i\}) + \sum_{R \subseteq N \setminus \{i,j\}} [M^\Phi]_{i,R \cup \{i,j\}}v(R \cup \{i, j\})$$

$$+ \sum_{R \subseteq N \setminus \{i,j\}} [M^\Phi]_{i,R \cup \{j\}}v(R \cup \{j\}) + \sum_{\substack{R \subseteq N \setminus \{i,j\} \\ R \neq \emptyset}} [M^\Phi]_{i,R}v(R)$$

$$- \sum_{R \subseteq N \setminus \{i,j\}} [M^\Phi]_{j,R \cup \{j\}}(\pi_{ij}v)(R \cup \{j\}) - \sum_{R \subseteq N \setminus \{i,j\}} [M^\Phi]_{j,R \cup \{i,j\}}(\pi_{ij}v)(R \cup \{i, j\})$$

$$- \sum_{R \subseteq N \setminus \{i,j\}} [M^\Phi]_{j,R \cup \{i\}}(\pi_{ij}v)(R \cup \{i\}) - \sum_{\substack{R \subseteq N \setminus \{i,j\} \\ R \neq \emptyset}} [M^\Phi]_{j,R}(\pi_{ij}v)(R)$$

$$= \sum_{R \subseteq N \setminus \{i,j\}} \left([M^\Phi]_{i,R \cup \{i\}} - [M^\Phi]_{j,R \cup \{j\}} \right) v(R \cup \{i\})$$

$$+ \sum_{R \subseteq N \setminus \{i,j\}} \left([M^\Phi]_{i,R \cup \{i,j\}} - [M^\Phi]_{j,R \cup \{i,j\}} \right) v(R \cup \{i,j\})$$

$$+ \sum_{R \subseteq N \setminus \{i,j\}} \left([M^\Phi]_{i,R \cup \{j\}} - [M^\Phi]_{j,R \cup \{i\}} \right) v(R \cup \{j\})$$

$$+ \sum_{\substack{R \subseteq N \setminus \{i,j\} \\ R \neq \emptyset}} \left([M^\Phi]_{i,R} - [M^\Phi]_{j,R} \right) v(R).$$

By symmetry property, $\Phi_i(v) - \Phi_{\pi_{ij}(i)}(\pi_{ij}v) = 0$. Note that the game $\langle N, v \rangle$ is arbitrary and so, the worths $v(S)$, $S \in \Omega$, can be chosen arbitrarily. Therefore, for all $R \subseteq N \setminus \{i,j\}$, we have

$$[M^\Phi]_{i,R \cup \{i\}} = [M^\Phi]_{j,R \cup \{j\}}, \quad [M^\Phi]_{i,R \cup \{j\}} = [M^\Phi]_{j,R \cup \{i\}}; \qquad (2.2.6)$$

and $\quad [M^\Phi]_{i,R \cup \{i,j\}} = [M^\Phi]_{j,R \cup \{i,j\}}, \quad [M^\Phi]_{i,R} = [M^\Phi]_{j,R}, \ (R \neq \emptyset). \qquad (2.2.7)$

For any $S, T \in \Omega$ with equal sizes $s = t$, we have $|S \setminus T| = |T \setminus S|$. Consider the case that $i \in S, j \in T$. If $S = T$, then by (2.2.6), we know that $[M^\Phi]_{i,S} = [M^\Phi]_{j,T}$. Otherwise, let $S \setminus T = \{i_1, i_2, \cdots, i_k\}$, $T \setminus S = \{j_1, j_2, \cdots, j_k\}$, and $i_1 = i$, $j_k = j$. Let permutation π be the composition of interchanges $\pi_{i_1 j_1}$, $\pi_{i_2 j_2}, \cdots, \pi_{i_k j_k}$. By (2.2.6) and (2.2.7), we have

$$[M^\Phi]_{i_1,S} = [M^\Phi]_{j_1, S \cup \{j_1\} \setminus \{i_1\}} = \cdots = [M^\Phi]_{j_k, S \cup \{j_1, j_2, \cdots, j_k\} \setminus \{i_1, i_2, \cdots, i_k\}}.$$

That is, $[M^\Phi]_{i,S} = [M^\Phi]_{j,T}$. Particularly, if $i = j$, then by another player $k \in N \setminus \{i\}$ and a coalition $R \in \Omega, R \ni k$ such that $r = s = t$, we know that $[M^\Phi]_{i,S} = [M^\Phi]_{k,R} = [M^\Phi]_{i,T}$. Similarly, if $i \notin S, j \notin T$, we can prove that $[M^\Phi]_{i,S} = [M^\Phi]_{j,T}$. $\qquad \Box$

From this, for a linear, symmetric value Φ, of which the representation matrix is M^Φ, each entry $[M^\Phi]_{i,S}$ of M^Φ is only related to the size s of the coalition S and the membership or nonmembership between the player i and the coalition S. Therefore, we denote the entry $[M^\Phi]_{i,S}$ as m_s^Φ for $i \in S$, otherwise as $m_{s^-}^\Phi$, for all $S \in \Omega$.

Corollary 2.2.6. *Any linear and symmetric value Φ on $\mathcal{G}^N$ can be expressed as*

$$\Phi_i(v) = \sum_{S \ni i} m_s^\Phi v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s-}^\Phi v(S), \quad \text{for all games } \langle N, v \rangle, \text{ and all } i \in N.$$

Furthermore, we have the following theorem.

Theorem 2.2.7. *Let Φ be a linear, symmetric value on $\mathcal{G}^N$. Then Φ is efficient if and only if the representation matrix M^Φ satisfies*

$$m_{s-}^\Phi = -\frac{s}{n-s} m_s^\Phi, \text{ for all } 1 \leq s < n, \text{ and } m_n^\Phi = \frac{1}{n}. \tag{2.2.8}$$

Proof. Let Φ be a linear, symmetric value on $\mathcal{G}^N$ with the representation matrix M^Φ. By Corollary 2.2.6, for any game $\langle N, v \rangle$, we have

$$\sum_{i \in N} \Phi_i(v) = \sum_{i \in N} \sum_{S \in \Omega} [M^\Phi]_{i,S} v(S) = \sum_{S \in \Omega} \Big(\sum_{i \in N} [M^\Phi]_{i,S} \Big) v(S)$$

$$= \sum_{S \in \Omega} \Big[\sum_{i \in S} m_s^\Phi + \sum_{i \notin S} m_{s-}^\Phi \Big] v(S)$$

$$= \sum_{S \in \Omega} \Big[s m_s^\Phi + (n-s) m_{s-}^\Phi \Big] v(S).$$

Together with efficiency, we have $\sum\limits_{S \in \Omega} \Big[s m_s^\Phi + (n-s) m_{s-}^\Phi \Big] v(S) = v(N)$. Note that the game $\langle N, v \rangle$ is arbitrary and so, the worths $v(S)$, $S \in \Omega$, can be chosen arbitrarily. Thus, $s m_s^\Phi + (n-s) m_{s-}^\Phi = 0$, for all $S \in \Omega$, $S \neq N$, and $n m_n^\Phi = 1$ for $S = N$. That is,

$$m_{s-}^\Phi = -\frac{s}{n-s} m_s^\Phi, \text{ for all } 1 \leq s < n, \text{ and } m_n^\Phi = \frac{1}{n}. \qquad \square$$

By this theorem and Corollary 2.2.6, we can get the formula given by Ruiz *et al.* [73] for the class of linear, symmetric and efficient values.

Corollary 2.2.8 (Ruiz *et al.* [73]). *A value Φ on $\mathcal{G}^N$ possesses linearity, symmetry and efficiency if and only if there exists $m_s^\Phi, s = 1, 2, \cdots, n-1$, such that for any game $\langle N, v \rangle$,*

$$\Phi(v) = M^\Phi \cdot v, \quad \text{where} \quad [M^\Phi]_{i,S} = \begin{cases} \frac{1}{n}, & \text{if } S = N; \\ m_s^\Phi, & \text{if } i \in S, S \neq N; \\ -\frac{s}{n-s} m_s^\Phi, & \text{if } i \notin S, \end{cases}$$

i.e.,

$$\Phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N \\ S \ni i}} m_s^\Phi v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} \frac{s}{n-s} m_s^\Phi v(S), \quad \text{for all } i \in N.$$

2.3   The Shapley standard matrix

In this section, we continue the matrix procedure as started in Section 2.2. We characterize the representation matrix of the Shapley value named Shapley standard matrix, by using the classic axioms: linearity, efficiency, symmetry and the null player property. Moreover, some other properties of the Shapley standard matrix are described.

Theorem 2.3.1. *Let $M^\Phi = \left[M^\Phi\right]_{i \in N, S \in \Omega}$ be the representation matrix of a linear value Φ on $\mathcal{G}^N$ possessing symmetry, efficiency and the dummy player property (i.e., the Shapley value). Then*

$$[M^\Phi]_{i,S} = \begin{cases} \frac{(s-1)!(n-s)!}{n!}, & \text{if } i \in S; \\ -\frac{s!(n-s-1)!}{n!}, & \text{if } i \notin S. \end{cases} \tag{2.3.1}$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ possessing symmetry, efficiency, the dummy player property and M^Φ be its representation matrix. For any player $i \in N$, we consider a game $\langle N, v \rangle$ as follows. For all $S \in \Omega, S \not\ni i$, the worths $v(S)$ and $v(\{i\})$ are chosen arbitrarily, then let $v(S \cup \{i\}) = v(S) + v(\{i\})$, to ensure that i is a dummy player in $\langle N, v \rangle$. By Corollary 2.2.6, we have

$$\begin{aligned}
\Phi_i(v) &= \sum_{S \ni i} m_s^\Phi v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s^-}^\Phi v(S) \\
&= \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s+1}^\Phi v(S \cup \{i\}) + m_1^\Phi v(\{i\}) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s^-}^\Phi v(S) \\
&= \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s+1}^\Phi v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s+1}^\Phi v(\{i\}) + m_1^\Phi v(\{i\}) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s^-}^\Phi v(S) \\
&= \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} \left[m_{s+1}^\Phi + m_{s^-}^\Phi\right] v(S) + \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s+1}^\Phi v(\{i\}) + m_1^\Phi v(\{i\}).
\end{aligned}$$

By the dummy player property, $\Phi_i(v) = v(\{i\})$. Note that the worths $v(S)$, $S \in \Omega, S \not\ni i$ can be arbitrary values. Particularly, consider the unanimity games $\langle N, u_S \rangle$, for all $S \in \Omega, S \not\ni i$. We conclude that

$$m_{s+1}^{\Phi} + m_{s-}^{\Phi} = 0, \quad \text{for all } s = 1, 2, \cdots, n-1.$$

By (2.2.8) in Theorem 2.2.7, we have $m_n^{\Phi} = \frac{1}{n}$ as well as

$$m_{s+1}^{\Phi} = -m_{s-}^{\Phi} = \frac{s}{n-s} m_s^{\Phi}, \quad \text{for all } s = 1, 2, \cdots, n-1. \tag{2.3.2}$$

We obtain $m_{s+1}^{\Phi}, m_{s-}^{\Phi}$ for all $s = n-1, n-2, \cdots, 1$ recursively as follows.

$$\begin{cases} m_s^{\Phi} = \frac{(s-1)!(n-s)!}{n!}, \\ m_{s-}^{\Phi} = -\frac{s!(n-s-1)!}{n!}; \end{cases} \quad i.e., \ [M^{\Phi}]_{i,S} = \begin{cases} \frac{(s-1)!(n-s)!}{n!}, & \text{if } i \in S; \\ -\frac{s!(n-s-1)!}{n!}, & \text{if } i \notin S. \end{cases}$$

What is more,

$$\Phi_i(v) = \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} m_{s+1}^{\Phi} v(\{i\}) + m_1^{\Phi} v(\{i\}) = v(\{i\}) \sum_{s=1}^{n-1} \binom{n-1}{s} m_{s+1}^{\Phi} + \frac{1}{n} v(\{i\})$$

$$= v(\{i\}) \sum_{s=1}^{n-1} \frac{1}{n} + \frac{1}{n} v(\{i\}) = v(\{i\}). \qquad \square$$

We denote by M^{Sh} the representation matrix of the Shapley value on $\mathcal{G}^N$, and we call it the *Shapley standard matrix* (Xu, Driessen and Sun [101]). We restate the Shapley value in terminology of the Shapley standard matrix as follows.

Definition 2.3.2. *Given any game* $\langle N, v \rangle$, *the Shapley value* $Sh(v)$ *is represented by the Shapley standard matrix* M^{Sh} *as:*

$$Sh(v) = M^{Sh} \cdot v,$$

where the Shapley standard matrix $M^{Sh} = \left[M^{Sh} \right]_{i \in N, S \in \Omega}$ *is given by* (2.3.1).

The columns of the Shapley standard matrix M^{Sh} possess the following *anti-complementarity property*.

Proposition 2.3.3. *Let* $\left[M^{Sh}\right]_T$ *be the column of* M^{Sh} *indexed by coalition* $T \in \Omega$. *Then it holds that* $\left[M^{Sh}\right]_S = -\left[M^{Sh}\right]_{N\setminus S}$, *for all* $S \in \Omega$, $S \neq N$. *Consequently,* $\sum\limits_{S\in\Omega} \left[M^{Sh}\right]_S = \left[M^{Sh}\right]_N$.

Proof. For any coalition $S \in \Omega, S \neq N$, it is sufficient to show that

$$\left[M^{Sh}\right]_{i,S} + \left[M^{Sh}\right]_{i,N\setminus S} = 0, \quad \text{for all } i \in N.$$

Without loss of generality, suppose that $i \in S$. By the definition of the Shapley standard matrix M^{Sh}, we conclude that

$$\left[M^{Sh}\right]_{i,S} + \left[M^{Sh}\right]_{i,N\setminus S} = \frac{(s-1)!(n-s)!}{n!} - \frac{(n-s)!(s-1)!}{n!} = 0. \quad \square$$

The anti-complementarity property of the Shapley standard matrix implies an alternative formula for the Shapley value, due to Driessen [24].

Corollary 2.3.4 (Driessen [24]). *The Shapley value* $Sh(v)$ *of a game* $\langle N, v\rangle$ *is of the following form:*

$$Sh_i(v) = \sum_{S\ni i} \frac{(s-1)!(n-s)!}{n!}\Big[v(S) - v(N\setminus S)\Big], \quad \text{for all } i \in N.$$

Proof by the matrix approach. For any game $\langle N, v\rangle$, since $v(\emptyset) = 0$, for all $i \in N$, by Proposition 2.3.3, we have

$$Sh_i(v) = \sum_{S\in\Omega}[M^{Sh}]_{i,S}v(S) = \sum_{S\ni i}[M^{Sh}]_{i,S}v(S) + \sum_{\substack{S\not\ni i\\S\neq\emptyset}}[M^{Sh}]_{i,S}v(S)$$

$$= \sum_{S\ni i}[M^{Sh}]_{i,S}v(S) + \sum_{\substack{T\ni i\\T\neq N}}[M^{Sh}]_{i,N\setminus T}v(N\setminus T)$$

$$= \sum_{S\ni i}[M^{Sh}]_{i,S}v(S) - \sum_{T\ni i}[M^{Sh}]_{i,T}v(N\setminus T)$$

$$= \sum_{S\ni i}\Big\{[M^{Sh}]_{i,S}v(S) - [M^{Sh}]_{i,S}v(N\setminus S)\Big\}$$

$$= \sum_{S\ni i}[M^{Sh}]_{i,S}\big[v(S) - v(N\setminus S)\big]$$

$$= \sum_{S\ni i}\frac{(s-1)!(n-s)!}{n!}\big[v(S) - v(N\setminus S)\big]. \quad \square$$

The self-duality of the Shapley value can be translated into matrix interpretation as follows.

Proposition 2.3.5. *The Shapley standard matrix M^{Sh} satisfies $M^{Sh}Q = M^{Sh}$. That is to say, the Shapley value satisfies the self-duality property in that the Shapley values of the initial game and its dual game are equal.*

Proof by the matrix approach. It is sufficient to check the column equalities $\left[M^{Sh}Q\right]_T = \left[M^{Sh}\right]_T$ for all coalitions $T \in \Omega$. Due to the algebraic representation of a column of a matrix product, it holds that

$$\left[M^{Sh}Q\right]_T = \sum_{S \in \Omega}[Q]_{S,T}[M^{Sh}]_S$$

By (2.1.4) and Proposition 2.3.3, we obtain the following. If $T \neq N$, then

$$\left[M^{Sh}Q\right]_T = [Q]_{N\setminus T,T}[M^{Sh}]_{N\setminus T} = -[M^{Sh}]_{N\setminus T} = [M^{Sh}]_T.$$

If $T = N$, then

$$\left[M^{Sh}Q\right]_N = \sum_{S \in \Omega}[Q]_{S,N}\left[M^{Sh}\right]_S = \sum_{S \in \Omega}\left[M^{Sh}\right]_S = \left[M^{Sh}\right]_N.$$

That is to say, for a game $\langle N, v \rangle$ and its dual game $\langle N, v^* \rangle$, we have

$$Sh(v^*) = M^{Sh} \cdot v^* = M^{Sh} \cdot Q \cdot v = M^{Sh} \cdot v = Sh(v). \qquad \square$$

To conclude with, we show that the Shapley standard matrix is full row rank.

Proposition 2.3.6. *The rank of the Shapley standard matrix M^{Sh} with respect to the game space $\mathcal{G}^N$ satisfies $rank(M^{Sh}) = n$.*

Proof. Obviously, $rank(M^{Sh}) \leq n$ because of n rows. Consider the columns of M^{Sh} indexed by single player coalitions and the grand coalition:

$$M^{Sh} = \begin{pmatrix} \frac{1}{n} & -\frac{1}{n(n-1)} & \cdots & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \\ -\frac{1}{n(n-1)} & \frac{1}{n} & \cdots & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ -\frac{1}{n(n-1)} & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

Adding the multiplication of last column by $\frac{1}{n-1}$ to all of columns indexed by single player coalitions, we get

$$
\begin{pmatrix}
\frac{1}{n-1} & 0 & \cdots & 0 & \cdots & \frac{1}{n} \\
0 & \frac{1}{n-1} & \cdots & 0 & \cdots & \frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \frac{1}{n-1} & \cdots & \frac{1}{n}
\end{pmatrix}.
$$

Because of there are n linear independent columns, $\mathrm{rank}(M^{Sh}) \geq n$. Therefore, $\mathrm{rank}(M^{Sh}) = n$. □

Without proof, we emphasize that for every permutation $\pi \in \Pi^N$, the set of columns $[M^{Sh}]_{\pi(\{1\})}, [M^{Sh}]_{\pi(\{1,2\})}, \cdots, [M^{Sh}]_{\pi(N)}$ of M^{Sh} are maximal linearly independent.

Chapter 3

Two types of associated consistency for the Shapley value

The aim of this chapter is to develop the matrix approach for the Shapley value. Two types of associated consistency are analyzed for characterizing the Shapley value.

In view of Hamiache's associated game is a linear operator on the game space. The corresponding square-coalitional matrix called the associated transformation matrix M_λ is introduced and the matrix equality $M^{Sh} = M^{Sh} M_\lambda$ implies that the Shapley value is consistent under this linear operator. Motivated by the Shapley value is also consistent under the dual operator, to combine the dual operator Q and the linear operator M_λ for the associated game in terms of the similarity property for matrices, we introduce a new linear operator called the dual similar associated game with the transformation matrix $M_\lambda^D = Q M_\lambda Q^{-1}$. The diagonalization procedure of M_λ, M_λ^D and the inessential property for coalitional matrices are fundamental tools to prove the convergence of the sequence of repeated (dual similar) associated games as well as its limit game to be inessential. The similarity of matrices transfer associated consistency into dual similar associated consistency. The Shapley value is axiomatized as the unique value satisfying the inessential game property, continuity and (dual similar) associated consistency.

3.1   Introduction

Hamiache's recent axiomatization of the Shapley value states that the Shapley value is the unique value satisfying the inessential game property, continuity and associated consistency (see [37]). In his paper, an associated game $\langle N, v_\lambda^{Sh} \rangle$ is constructed. And a sequence of games is defined, where the term of order m, in this sequence, is the associated game of the term of order $m - 1$. He shows that this sequence of games converges and that the limit game is inessential. The value is obtained by using the inessential game property, the associated consistency and the continuity axioms. As a by-product, neither the linearity nor the efficiency axioms are needed. The uniqueness proof in Hamiache's axiomatization is rather difficult and full of combinatorial calculations.

At the second World Congress of the Game Theory Society held at Marseille, France (July 2004), Hamiache suggested a matrix approach to his axiomatization of the Shapley value. In Hamiache's paper [38], Hamiache introduces a matrix formula of the associated game and studies the diagonalizability property of the representation matrix, then presents a new (existence) proof, based on basic linear algebra, of his axiomatization of the Shapley value. Its uniqueness proof mainly relies on the existence of the limit game of the sequence of the associated games, and the link of his axioms with the eldest axiomatization of the Shapley value through efficiency, additivity, the null player property, and equal treatment property. It motivates our research and aim to offer the algebraic representation and the matrix approach for the axiomatization of the Shapley value.

In this chapter, the matrix approach is adopted to develop Hamiache's axiomatization of the Shapley value. In Section 3.2, both the Shapley value and the associated game are represented algebraically by their coalitional matrices called the Shapley standard matrix M^{Sh} and the associated transformation matrix M_λ, respectively. By the diagonalization procedure of M_λ, we derive the convergence of the sequence of repeated associated games. The limit game is shown to be inessential in terms of the inessential property of coalitional matrices. In Section 3.3, the associated consistency for the Shapley value is formulated as the matrix equality $M^{Sh} = M^{Sh} M_\lambda$. We achieve a matrix approach for Hamiache's axiomatization of the Shapley value.

Then we construct the dual similar associated game and introduce the

dual similar associated transformation matrix M_λ^D as well in Section 3.4. In the game-theoretic framework we show that the dual game of the dual similar associated game is Hamiache's associated game of the dual game. For the purpose of matrix analysis, we derive the similarity relationship $M_\lambda^D = Q M_\lambda Q^{-1}$ between the dual similar associated transformation matrix M_λ^D and the associated transformation matrix M_λ, where the transformation matrix Q represents the dual operator on games. It yields the inessential property for the limit game of the convergent sequence of repeated dual similar associated games. In Section 3.5, we show that this similarity of matrices also transfers associated consistency into dual similar associated consistency. Finally, we axiomatize the Shapley value as the unique value satisfying the inessential game property, continuity and dual similar associated consistency.

3.2 The associated transformation matrix

Firstly, let us recite the definition of the associated game. Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, Hamiache [37] define its *associated game* $\langle N, v_\lambda^{Sh} \rangle$ as follows:

$$v_\lambda^{Sh}(S) = v(S) + \lambda \sum_{j \in N \setminus S} \left[v(S \cup \{j\}) - v(S) - v(\{j\}) \right], \quad \text{for all} \ \ S \in \Omega.$$

The associated game may be considered as an adaptation of a given game such that it reflects an optimistic self-evaluation of worths of coalitions[1]. Notice that $v_\lambda^{Sh}(N) = v(N)$ and moreover, $v_\lambda^{Sh} = v$ for all inessential games $\langle N, v \rangle$. We do not care about the trivial case $\lambda = 0$. Obviously, the worth $v_\lambda^{Sh}(S)$ of coalition S can be expressed as

$$v_\lambda^{Sh}(S) = \left[1 - (n - s)\lambda \right] v(S) + \lambda \sum_{j \in N \setminus S} v(S \cup \{j\}) - \lambda \sum_{j \in N \setminus S} v(\{j\}).$$

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[1]To be coherent with Hamiache's ([37], pages 281-282) myopic vision of the environment, every coalition S ignores the links existing between players in $N \setminus S$. As a consequence, a coalition S considers itself at the center of a star-like graph, which is equivalent to say that coalition S considers players in $N \setminus S$ as isolated elements. Following the additional "divide and rule" behavior, coalition S may believe that the appropriation of at least a part of the surpluses $v(S \cup \{j\}) - v(S) - v(\{j\})$, generated by its cooperation with each one of the isolated players $j \in N \setminus S$, is within reach. Thus coalition S may evaluate its own worth, $v_\lambda^{Sh}(S)$, as the sum of its worth in the original game, $v(S)$, and of a given percentage $\lambda \in [0, 1]$, of all the possible previous surpluses.

Clearly, the associated game is a linear mapping on the game space $\mathcal{G}^N$ to itself. In order to study this linear operator by the matrix approach, as suggested by Hamiache [38], we introduce the associated transformation matrix to represent the associated game and the sequence of repeated associated games as follows.

Definition 3.2.1. *Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, the associated game $\langle N, v_\lambda^{Sh} \rangle$ is represented by the associated transformation matrix M_λ as*

$$v_\lambda^{Sh} = M_\lambda \cdot v,$$

where the matrix $M_\lambda = \left[M_\lambda \right]_{S,T \in \Omega}$ is square-coalitional defined by

$$\left[M_\lambda \right]_{S,T} = \begin{cases} 1 - (n-s)\lambda, & \text{if } T = S; \\ \lambda, & \text{if } T = S \cup \{j\} \text{ and } j \in N \setminus S; \\ -\lambda, & \text{if } T = \{j\} \text{ and } j \in N \setminus S; \\ 0, & \text{otherwise.} \end{cases}$$

And the sequence of repeated associated games $\{ \langle N, v_\lambda^{m*Sh} \rangle \}_{m=0}^{\infty}$ is defined recursively as

$$v_\lambda^{m*Sh} = M_\lambda \cdot v_\lambda^{(m-1)*Sh}, \quad \text{for all } m \geq 1, \quad \text{where } v_\lambda^{0*Sh} = v.$$

Now the main goal is to investigate eigenvalues and eigenvectors of the associated transformation matrix M_λ. According to the concept of inessential game, we introduce a corresponding property for the coalitional matrix.

Definition 3.2.2. *A row-coalitional matrix $M = [\vec{M}_S]_{S \in \Omega}$ is called row-inessential if its row $\vec{M}_S$ indexed by coalition S, satisfies*

$$\vec{M}_S = \sum_{i \in S} \vec{M}_i, \quad \text{for all } S \in \Omega.$$

For simplicity, $\vec{M}_i$ denotes the row $\vec{M}_{\{i\}}$, for all $i \in N$.

Proposition 3.2.3. *1 is an eigenvalue of M_λ, the eigenvectors corresponding to eigenvalue 1 are row-inessential, and the dimension of the eigenspace of eigenvalue 1 is equal to n.*

Proof. Since $v_\lambda^{Sh}(N) = v(N)$, the last row of matrix $I - M_\lambda$ is the zero-vector. So 1 is an eigenvalue of M_λ. Let $\vec{x} = (x_S)_{S \in \Omega}$ be an eigenvector corresponding

to eigenvalue 1. Interpret $\vec{x}$ as a row-coalitional matrix. Since $(I - M_\lambda)\vec{x} = \vec{0}$, we have

$$(n - s)x_S - \sum_{j \in N \setminus S} x_{S \cup \{j\}} + \sum_{j \in N \setminus S} x_j = 0, \quad \text{for all } S \in \Omega.$$

By this equation, for $n - s = 1$ and $N \setminus S = \{j\}$, we have

$$x_{N \setminus \{j\}} + x_j = x_N, \quad \text{for all } j \in N.$$

By induction on $n - s = |N \setminus S|$, we obtain that

$$x_S + \sum_{j \in N \setminus S} x_j = x_N, \quad \text{for all } S \in \Omega.$$

For S being the case of single player coalition, it holds that $x_N = \sum_{j \in N} x_j$. So we conclude that

$$x_S = \sum_{j \in S} x_j, \quad \text{for all } S \in \Omega.$$

From the inessential property of any eigenvector $\vec{x}$ corresponding to eigenvalue 1, it follows immediately that the dimension of the eigenspace of eigenvalue 1 is n. □

Proposition 3.2.4. *For every k ($2 \leq k \leq n$), we have*

$$rank[(1 - k\lambda)I - M_\lambda] \leq 2^n - 1 - \binom{n}{k},$$

and hence, $1 - k\lambda$ is an eigenvalue of M_λ.

Proof. For any k ($2 \leq k \leq n$), let $\vec{x} = (x_S)_{S \in \Omega}$ be a vector such that

$$[(1 - k\lambda)I - M_\lambda]\vec{x} = \vec{0}.$$

Then the following system of linear equations holds,

$$(n - s - k)x_S - \sum_{j \in N \setminus S} x_{S \cup \{j\}} + \sum_{j \in N \setminus S} x_j = 0, \quad \text{for all } S \in \Omega. \qquad (3.2.1)$$

For the case that $S = N$, since $kx_N = 0$ and $k \neq 0$, we have $x_N = 0$. In the sequel, we show that, for any k, there are $\binom{n}{k}$ identical equations in the linear system of $[(1 - k\lambda)I - M_\lambda]\vec{x} = \vec{0}$.

If $s = n - 1$ and $S = N \setminus \{j\}$, by (3.2.1) we have

$$(1 - k)x_{N \setminus \{j\}} - x_N + x_j = 0.$$

That is

$$x_{N \setminus \{j\}} = \frac{1}{k - 1} x_j, \quad \text{for all } j \in N. \tag{3.2.2}$$

We use induction on $n - s$ such that $n - s < k$ to show that

$$x_S = \frac{1}{k - 1} \sum_{j \in N \setminus S} x_j, \quad \text{for all } S \in \Omega, S \neq N. \tag{3.2.3}$$

Now suppose (3.2.3) is true for all $n - s \leq t - 1$, where $t < k$. For the case that $n - s = t$, let $S = N \setminus T$. By (3.2.1), we have

$$(t - k)x_{N \setminus T} - \sum_{i \in T} x_{(N \setminus T) \cup \{i\}} = -\sum_{j \in T} x_j.$$

By the induction hypothesis and $i \in T$, we obtain

$$x_{(N \setminus T) \cup \{i\}} = \frac{1}{k - 1} \Big(\sum_{j \in T} x_j - x_i \Big).$$

Thus

$$(t - k)x_{N \setminus T} - \frac{t}{k - 1} \sum_{j \in T} x_j + \frac{1}{k - 1} \sum_{i \in T} x_i = -\sum_{j \in T} x_j.$$

$$(t - k)x_{N \setminus T} = \frac{t - k}{k - 1} \sum_{j \in T} x_j. \tag{3.2.4}$$

So, if $t < k$, then (3.2.4) implies that (3.2.3) holds for $s = n - t$ and $n - s < k$.

Furthermore, if we start the above calculations with the conclusion that (3.2.3) holds for the case $t = k - 1$, we reach (3.2.4) being true for the case $t = k$. Then $\binom{n}{k}$ linear equations in $[(1 - k\lambda)I - M_\lambda]\vec{x} = \vec{0}$ are identical equations. Hence,

$$\text{rank}[(1 - k\lambda)I - M_\lambda] \leq 2^n - 1 - \binom{n}{k}.$$

Consequently, $1 - k\lambda$ is an eigenvalue of M_λ, for $2 \leq k \leq n$.   □

Here we recall some results in algebra theory for deriving more properties of the associated transformation matrix M_λ.

Lemma 3.2.5 (Algebra results [54]). *Let A be a square matrix of order p.*

1. *The dimension d of the solution space of the linear system of equations $A\vec{x} = \vec{0}$ satisfies $d = p - rank(A)$.*

2. *For every eigenvalue of matrix A, its (algebraic) multiplicity is at least the dimension of the corresponding eigenspace.*

3. *The sum of the multiplicities of all eigenvalues of matrix A equals its order p.*

4. *The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals p, and this happens if and only if the dimension of the eigenspace for each eigenvalue equals the multiplicity of the eigenvalue.*

Theorem 3.2.6. *Eigenvalues of the associated transformation matrix M_λ are $1, 1 - k\lambda$, $k = 2, 3, \cdots, n$, and multiplicities corresponding to these eigenvalues are $\binom{n}{1}, \binom{n}{k}, k = 2, 3, \cdots, n$.*

Proof. Let $u_1 = 1$ and $u_k = 1 - k\lambda$ ($2 \le k \le n$). By Proposition 3.2.3 and 3.2.4, we know that u_k ($2 \le k \le n$) are eigenvalues of M_λ. Let d_k denote the dimension of the eigenspace corresponding to $(u_k I - M_\lambda)\vec{x} = \vec{0}$. By Proposition 3.2.3, we obtain $d_1 = n$, whereas from Proposition 3.2.4 and Lemma 3.2.5 (1), we derive that

$$d_k = 2^n - 1 - \mathrm{rank}(u_k I - M_\lambda) \ge \binom{n}{k}, \quad k = 2, 3, \cdots, n.$$

Since the multiplicity m_k of eigenvalue u_k satisfies $m_k \ge d_k$, we have

$$2^n - 1 = \sum_{k=1}^{n} m_k \ge n + \sum_{k=2}^{n} d_k \ge \binom{n}{1} + \sum_{k=2}^{n} \binom{n}{k} = 2^n - 1.$$

Thus $m_k = d_k = \binom{n}{k}$, for all $1 \le k \le n$ and matrix M_λ has no other eigenvalues. $\qquad\square$

From Theorem 3.2.6, we conclude that the matrix M_λ is diagonalizable. To prove the next theorem, we make use of the following properties of row-coalitional matrices.

Lemma 3.2.7. *Let M be a row-coalitional matrix and A be a matrix.*

1. *If M is row-inessential, then the row-coalitional matrix MA is row-inessential.*

2. *If A is invertible, then MA is row-inessential if and only if M is row-inessential.*

3. *For all games $\langle N, v \rangle$, if M is row-inessential, then the game $\langle N, M \cdot v \rangle$ is inessential.*

Proof. Write $M = [\vec{M}_S]_{S \in \Omega}$, where $\vec{M}_S$ is the row of M indexed by a coalition S, $S \in \Omega$.

1. By Definition 3.2.2, if M is row-inessential, *i.e.*, $\vec{M}_S = \sum\limits_{i \in S} \vec{M}_i$, then for any $S \in \Omega$,

$$[\vec{MA}]_S = \vec{M}_S A = (\sum_{i \in S} \vec{M}_i) A = \sum_{i \in S} (\vec{M}_i A) = \sum_{i \in S} [\vec{MA}]_i.$$

Thus, MA is row-inessential.

2. If A is invertible, then for all $S \in \Omega$,

$$\vec{M}_S A = \sum_{i \in S} (\vec{M}_i A) = (\sum_{i \in S} \vec{M}_i) A \quad \text{if and only if} \quad \vec{M}_S = \sum_{i \in S} \vec{M}_i.$$

By the conclusion 1, we have MA is row-inessential if and only if M is row-inessential.

3. If M is row-inessential, then for every game $\langle N, v \rangle$, and all $S \in \Omega$, we have $\vec{M}_S \cdot v = (\sum\limits_{i \in S} \vec{M}_i) \cdot v = \sum\limits_{i \in S} (\vec{M}_i \cdot v)$. Therefore, $M \cdot v = [\vec{M}_S \cdot v]_{S \in \Omega}$ is a inessential game. $\square$

Now we present the following important properties of the associated transformation matrix M_λ by its diagonalization procedure and Proposition 3.2.3.

Lemma 3.2.8. *Let M_λ be the associated transformation matrix.*

1. *$M_\lambda = P D_\lambda P^{-1}$, where P consists of eigenvectors of M_λ corresponding to eigenvalues $1, 1 - k\lambda$ $(2 \le k \le n)$ and*

$$D_\lambda = diag(\underbrace{1, \cdots, 1}_{\binom{n}{1} \ times}, \underbrace{1 - 2\lambda, \cdots, 1 - 2\lambda}_{\binom{n}{2} \ times}, \cdots, \underbrace{1 - n\lambda}_{\binom{n}{n} \ times}).$$

2. *If $0 < \lambda < \frac{2}{n}$, then $\lim_{m \to \infty} (M_\lambda)^m = PDP^{-1}$, where*

$$D = diag(\underbrace{1, \cdots, 1}_{n \ times}, \underbrace{0, \cdots, 0}_{2^n - 1 - n \ times}).$$

3. *The row-coalitional matrix PD equals*

$$PD = [\vec{x^1}, \vec{x^2}, \cdots, \vec{x^n}, \vec{0}, \cdots, \vec{0}],$$

and is row-inessential, where columns $\vec{x^i}, i = 1, 2, \cdots, n$ are different eigenvectors of M_λ corresponding to the eigenvalue 1 and column $\vec{0}$ is the zero-vector.

Using the previous results, we derive the following theorem about the convergence of the sequence of repeated associated games.

Theorem 3.2.9. *Let $0 < \lambda < \frac{2}{n}$. The sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh} \rangle\}_{m=0}^{\infty}$ converges to the game $\langle N, \tilde{v} \rangle$, where $\tilde{v} = PDP^{-1} \cdot v$. Furthermore, the limit game $\langle N, \tilde{v} \rangle$ is inessential.*

Proof. By Lemma 3.2.8 (2),

$$\lim_{m \to \infty} v_\lambda^{m*Sh} = \lim_{m \to \infty} (M_\lambda)^m \cdot v = PDP^{-1} \cdot v.$$

Due to Lemma 3.2.7 (3) and $\tilde{v} = PDP^{-1} \cdot v$, the game $\langle N, \tilde{v} \rangle$ is inessential whenever the matrix PDP^{-1} is row-inessential. By Lemma 3.2.8 (3), the matrix PD is row-inessential. Together with Lemma 3.2.7 (2), it follows that the matrix PDP^{-1} is row-inessential too. $\qquad\square$

Remark 3.2.10. *The limit game $\langle N, \tilde{v} \rangle$ of the sequence of repeated associated games merely depends on the game $\langle N, v \rangle$ as $\tilde{v} = PDP^{-1}v$. And for any player $i \in N$, the limit worth $\tilde{v}(\{i\})$ is just the inner product of the i-th row of PDP^{-1} and the column vector v.*

3.3 Associated consistency for the Shapley value

In this section, we apply the results from the previous section to develop a matrix approach for Hamiache's axiomatization [37] of the Shapley value. Firstly, we recall the axiom of associated consistency of a value Φ on $\mathcal{G}^N$ introduced by Hamiache.

- *Associated consistency* : for every game $\langle N, v \rangle$ and its associated game $\langle N, v_\lambda^{Sh} \rangle$, the value Φ satisfies $\Phi(v) = \Phi(v_\lambda^{Sh})$.

According to associated consistency, the value is invariant under the adaptation of the game into the associated game. Because payoffs to players neither increase nor decrease, an associated consistent rule neutralizes the possible effects of optimistic self-evaluation of worths of coalitions. In matrix theory, it turns out that the Shapley standard matrix M^{Sh} is invariant under multiplication with the associated transformation matrix M_λ.

Lemma 3.3.1 (Hamiache [37]). *The Shapley value verifies associated consistency, that is $M^{Sh} = M^{Sh} M_\lambda$.*

Proof. Since $Sh(v) = M^{Sh} \cdot v$ and $Sh(v_\lambda^{Sh}) = M^{Sh}(M_\lambda \cdot v)$, it suffices to check that $M^{Sh} = M^{Sh} M_\lambda$ for showing that the Shapley value satisfies the associated consistency, *i.e.*,

$$M^{Sh}(M_\lambda - I) = \vec{0}.$$

By the definition of M^{Sh} and M_λ, for all $i \in N$ and for all $T \in \Omega$, the entry $\left[M^{Sh}(M_\lambda - I) \right]_{i,T}$ is given as follows:

$$\left[M^{Sh}(M_\lambda - I) \right]_{i,T}$$
$$= \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \left[M_\lambda - I \right]_{S,T} - \sum_{\substack{S \ni i \\ S \neq \emptyset}} \frac{s!(n-s-1)!}{n!} \left[M_\lambda - I \right]_{S,T}.$$

For simplicity, we denote $\left[M^{Sh}(M_\lambda - I) \right]_{i,T}$ by a, for all $T \in \Omega, i \in T$. If $i \in T$ and $t \geq 2$, then

$$a = \frac{(t-1)!(n-t)!}{n!} \left[M_\lambda - I \right]_{T,T} - \frac{(t-1)!(n-t)!}{n!} \left[M_\lambda - I \right]_{T \setminus \{i\}, T}$$
$$\qquad + \sum_{j \in T \setminus \{i\}} \frac{(t-2)!(n-t+1)!}{n!} \left[M_\lambda - I \right]_{T \setminus \{j\}, T}$$
$$= \frac{(t-1)!(n-t)!}{n!} (t-n)\lambda - \frac{(t-1)!(n-t)!}{n!} \lambda$$
$$\qquad + \frac{(t-2)!(n-t+1)!}{n!} (t-1)\lambda$$
$$= \frac{(t-1)!(n-t)!}{n!} \left[-(n-t)\lambda + (n-t+1)\lambda - \lambda \right] = 0.$$

If $i \notin T$ and $t \geq 2$, then

$$a = -\Big\{ \frac{t!(n-t-1)!}{n!} \big[M_\lambda - I\big]_{T,T} + \sum_{j \in T} \frac{(t-1)!(n-t)!}{n!} \big[M_\lambda - I\big]_{T \backslash \{j\}, T} \Big\}$$

$$= -\Big\{ \frac{t!(n-t-1)!}{n!} \big[-(n-t)\lambda \big] + \frac{(t-1)!(n-t)!}{n!} t\lambda \Big\}$$

$$= -\Big\{ \frac{t!(n-t-1)!}{n!} \big[-(n-t)\lambda + (n-t)\lambda \big] \Big\} = 0.$$

If $T = \{i\}$, then

$$a = \frac{(1-1)!(n-1)!}{n!} \big[M_\lambda - I\big]_{i, \{i\}} - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} \frac{s!(n-s-1)!}{n!} \big[M_\lambda - I\big]_{S, \{i\}}$$

$$= \frac{(1-1)!(n-1)!}{n!} \big[-(n-1)\lambda \big] - \sum_{1 \leq s \leq n-1} \frac{s!(n-s-1)!}{n!} (-\lambda) \binom{n-1}{s}$$

$$= -\frac{n-1}{n}\lambda + \sum_{1 \leq s \leq n-1} \frac{1}{n}\lambda = 0.$$

If $T = \{j\}$ and $j \neq i$, then

$$a = \sum_{\substack{S \subseteq N \backslash \{j\} \\ S \ni i}} \frac{(s-1)!(n-s)!}{n!} \big[M_\lambda - I\big]_{S, \{j\}}$$

$$\quad - \Big\{ \frac{(n-1-1)!}{n!} \big[M_\lambda - I\big]_{j, \{j\}} + \sum_{\substack{S \subseteq N \backslash \{i,j\} \\ S \neq \emptyset}} \frac{s!(n-s-1)!}{n!} \big[M_\lambda - I\big]_{S, \{j\}} \Big\}$$

$$= \sum_{1 \leq s \leq n-1} \frac{(s-1)!(n-s)!}{n!} (-\lambda) \binom{n-2}{s-1}$$

$$\quad - \Big\{ \frac{(n-1-1)!}{n!} \big[(1-n)\lambda \big] + \sum_{1 \leq s \leq n-2} \frac{s!(n-s-1)!}{n!} (-\lambda) \binom{n-2}{s} \Big\}$$

$$= \sum_{1 \leq s \leq n-1} \frac{(n-s)}{n(n-1)} (-\lambda) + \frac{(n-1)!}{n!}\lambda - \sum_{1 \leq s \leq n-2} \frac{n-s-1}{n(n-1)} (-\lambda) = 0.$$

These four cases imply that $M^{Sh} = M^{Sh} M_\lambda$. $\qquad\square$

Theorem 3.3.2 (Hamiache [37]). *For $0 < \lambda < \frac{2}{n}$, the Shapley value is the unique value satisfying the associated consistency, continuity and the inessential game property.*

Proof by the matrix approach. Obviously, the Shapley value satisfies the inessential game property and continuity, and by Lemma 3.3.1, the Shapley value satisfies the associated consistency.

So, let us now turn to the uniqueness proof. Consider a value Φ satisfying these three axioms. Fix the game $\langle N, v \rangle$. We show that $\Phi(v) = Sh(v)$. By both the associated consistency and continuity of Φ, it holds that

$$\Phi(v) = \Phi(\tilde{v}), \quad \text{where } \tilde{v} = PDP^{-1} \cdot v.$$

Since the limit game $\langle N, \tilde{v} \rangle$ is shown to be inessential in Theorem 3.2.9, the inessential game property for Φ yields that $\Phi_i(\tilde{v}) = \tilde{v}(\{i\})$ for all $i \in N$. In summary, $\Phi(v) = (\tilde{v}(\{i\}))_{i \in N}$. Similarly, since the Shapley value also satisfies these three axioms, it follows that

$$Sh(v) = Sh(\tilde{v}) = (\tilde{v}(\{i\}))_{i \in N}.$$

From this, we conclude that $\Phi(v) = Sh(v)$. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.3.3. *Since $Sh(v) = M^{Sh} \cdot v$ and $\tilde{v} = PDP^{-1} \cdot v$, we deduce from $Sh(v) = (\tilde{v}(\{i\}))_{i \in N}$ that the Shapley standard matrix M^{Sh} is just the first part of the row-coalitional matrix PDP^{-1} indexed by single player coalitions. In fact, PDP^{-1} is the extension of M^{Sh} by the row inessential property.*

3.4   The dual similar associated game

Both the dual game and the associated game can be regarded as two linear operators on the game space $\mathcal{G}^N$. Moveover, the associated consistency and self-duality imply that the linear operator of the Shapley value is consistent under these linear operators with respect to the associated game and the dual game. Motivated by this, we intend to introduce a new type of linear operator on game space by combining these two operators in the framework of linear algebra.

Concerning the associated worth $v_\lambda^{Sh}(S)$, the net benefits $v(S \cup \{j\}) - v(S) - v(\{j\})$ arising from mutual cooperation among the coalition S itself and any of each isolated non-members $j \in N \setminus S$, measures the surplus of the coalitional marginal contribution $\nabla^v(S, j) = v(S \cup \{j\}) - v(S)$ over the individual worth $\nabla^v(\emptyset, j) = v(\{j\})$. Consider a new associated game by revaluing

the worth of coalition S based on the marginal contributions of its members. Any member $j \in S$ contributes $\nabla^v(S, j) = v(S) - v(S \setminus \{j\})$ to the formation of the coalition S itself, whereas its contribution to the formation of the grand coalition N equals $\nabla^v(N, j) = v(N) - v(N \setminus \{j\})$. The net benefits $(v(N) - v(N \setminus \{j\})) - (v(S) - v(S \setminus \{j\}))$ measures the gain (or loss, if it is negative) of the overall marginal contribution $\nabla^v(N, j)$ over the coalitional marginal contribution $\nabla^v(S, j)$ of the member $j \in S$.

Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, define its *dual similar associated game* $\langle N, v_\lambda^D \rangle$ as follows:

$$v_\lambda^D(S) = v(S) + \lambda \sum_{j \in S} \left[\big(v(N) - v(N \setminus \{j\})\big) - \big(v(S) - v(S \setminus \{j\})\big) \right], \quad \text{for all } S \in \Omega.$$

Notice that $v_\lambda^D(N) = v(N)$ and moreover, $v_\lambda^D = v$ for all inessential games $\langle N, v \rangle$. Similarly, for all $S \in \Omega$ we can express the worth $v_\lambda^D(S)$ as:

$$v_\lambda^D(S) = \big(1 - s\lambda\big)v(S) + s\lambda v(N) - \lambda \sum_{j \in S} v(N \setminus \{j\}) + \lambda \sum_{j \in S} v(S \setminus \{j\}).$$

By the matrix approach, we can define the dual similar associated game and the sequence of repeated dual similar associated games as follows.

Definition 3.4.1. *Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, its dual similar associated game $\langle N, v_\lambda^D \rangle$ is represented by the dual similar associated transformation matrix M_λ^D as*

$$v_\lambda^D = M_\lambda^D \cdot v,$$

where the matrix $M_\lambda^D = \big[M_\lambda^D\big]_{S,T \in \Omega}$ is square-coalitional defined by

$$\big[M_\lambda^D\big]_{S,T} = \begin{cases} 1 - s\lambda, & \text{if } T = S \text{ and } S \neq N; \\ s\lambda, & \text{if } T = N \text{ and } S \neq N; \\ \lambda, & \text{if } T = S \setminus \{j\}, j \in S \text{ and } S \neq N; \\ -\lambda, & \text{if } T = N \setminus \{j\}, j \in S \text{ and } S \neq N; \\ 1, & \text{if } T = S = N; \\ 0, & \text{otherwise.} \end{cases}$$

The sequence of repeated dual similar associated games $\{\langle N, v_\lambda^{m*D} \rangle\}_{m=0}^{\infty}$ is defined recursively as

$$v_\lambda^{m*D} = M_\lambda^D \cdot v_\lambda^{(m-1)*D}, \quad \text{for all } m \geq 1, \quad \text{where } v_\lambda^{0*D} = v.$$

Recall the concept of dual game in Section 1.2.1, its matrix restatement in Definition 2.1.3 and some properties of the dual matrix Q. We can conclude the following similarity relationship between the dual similar associated transformation matrix M_λ^D and the associated transformation matrix M_λ, as well as the corresponding relationship between these two types of associated games.

Lemma 3.4.2. *$M_\lambda^D = QM_\lambda Q$, or equivalently $QM_\lambda^D = M_\lambda Q$. In the game-theoretic context, the dual game of the dual similar associated game is the associated game of the dual game, i.e., $\langle N, (v_\lambda^D)^* \rangle = \langle N, (v^*)_\lambda^{Sh} \rangle$.*

Proof. For any row-coalitional matrix M, let $[M]_S$ denote the row of M indexed by any coalition $S \in \Omega$.

Since $[Q]_N = [M_\lambda]_N = [M_\lambda^D]_N = (0, \cdots, 0, 1)$, it is easy to check that $[QM_\lambda Q]_N = [M_\lambda^D]_N$. By the definitions of Q, M_λ and M_λ^D, for any coalition $S \in \Omega, S \neq N$, we have

$$
\begin{aligned}
[QM_\lambda Q]_S &= \sum_{T \in \Omega} [Q]_{S,T} [M_\lambda Q]_T = [M_\lambda Q]_N - [M_\lambda Q]_{N \setminus S} \\
&= \sum_{T \in \Omega} [M_\lambda]_{N,T} [Q]_T - \sum_{T \in \Omega} [M_\lambda]_{N \setminus S, T} [Q]_T \\
&= [Q]_N - \Big\{ (1 - s\lambda) [Q]_{N \setminus S} + \lambda \sum_{j \in S} [Q]_{(N \setminus S) \cup \{j\}} - \lambda \sum_{j \in S} [Q]_j \Big\} \\
&= [M_\lambda^D]_S.
\end{aligned}
$$

The latter row-vector equality has to be checked for all entries indexed by coalitions $T \in \Omega$.

Since $Q^2 = I$, so $QM_\lambda^D = M_\lambda Q$. That is to say, for any game $\langle N, v \rangle$, it holds that

$$
\langle N, (v_\lambda^D)^* \rangle = QM_\lambda^D \cdot v = M_\lambda Q \cdot v = \langle N, (v^*)_\lambda^{Sh} \rangle. \qquad \square
$$

The alternative proof. For any coalition $S \in \Omega$, by the definitions of the

dual game and the associated game, we have

$$\left((v^*)_\lambda{}^{Sh}\right)^*(S) = (v^*)_\lambda^{Sh}(N) - (v^*)_\lambda^{Sh}(N \setminus S)$$

$$= v^*(N) - \left\{\left(1 - s\lambda\right)v^*(N \setminus S) + \lambda \sum_{j \in S} v^*((N \setminus S) \cup \{j\}) - \lambda \sum_{j \in S} v^*(\{j\})\right\}$$

$$= v(N) - (1 - s\lambda)\left(v(N) - v(S)\right)$$

$$\qquad - \lambda \sum_{j \in S} \left[v(N) - v(S \setminus \{j\})\right] + \lambda \sum_{j \in S} \left[v(N) - v(N \setminus \{j\})\right]$$

$$= \left(1 - s\lambda\right)v(S) + s\lambda v(N) - \lambda \sum_{j \in S} v(N \setminus \{j\}) + \lambda \sum_{j \in S} v(S \setminus \{j\})$$

$$= v_\lambda^D(S). \qquad\qquad \square$$

Due to the duality property $Q^2 = I$, the above relationship can be recited as follows.

Corollary 3.4.3. $M_\lambda = Q M_\lambda^D Q$, *or equivalently* $Q M_\lambda = M_\lambda^D Q$. *In the game-theoretic context, the dual game of the associated game is the dual similar associated game of the dual game, i.e.,* $\langle N, (v_\lambda^{Sh})^* \rangle = \langle N, (v^*)_\lambda^D \rangle$.

The next diagram illustrates the commutative relationship between the two types of associated games in Lemma 3.4.2 and Corollary 3.4.3.

$$
\begin{array}{ccc}
\langle N, v \rangle & \xrightarrow[\text{associated game}]{\text{dual similar}} & \langle N, v_\lambda^D \rangle = \langle N, \left((v^*)_\lambda^{Sh}\right)^* \rangle \\[2em]
\text{dual} \Big\updownarrow \text{game} & & \text{dual} \Big\updownarrow \text{game} \\[2em]
\langle N, v^* \rangle & \xrightarrow[\text{game}]{\text{associated}} & \langle N, (v^*)_\lambda^{Sh} \rangle = \langle N, \left(v_\lambda^D\right)^* \rangle
\end{array}
$$

Diagram 3.4.1: The commutative relationship between
the two types of associated games

Recall the properties of the associated transformation matrix M_λ and the convergence of the sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh} \rangle\}_{m=0}^\infty$ in Section 3.2. Similar properties of the dual similar associated transformation matrix M_λ^D and the sequence of repeated dual similar associated games

$\{\langle N, v_\lambda^{m*D}\rangle\}_{m=0}^\infty$ can be derived. By Theorem 3.2.9, together with the relationship between the two types of associated games, we obtain the next theorem.

Theorem 3.4.4. *Let $0 < \lambda < \frac{2}{n}$, then the sequence of repeated dual similar associated games $\{\langle N, v_\lambda^{m*D}\rangle\}_{m=0}^\infty$ converges to the game $\langle N, \hat{v}\rangle$, where $\hat{v} = QPDP^{-1}Q \cdot v$. Furthermore, the limit game $\langle N, \hat{v}\rangle$ is inessential.*

Proof. By Lemma 3.4.2 and the duality $Q^2 = I$ in Proposition 2.1.7, we know

$$v_\lambda^{m*D} = (M_\lambda^D)^m \cdot v = (QM_\lambda Q)^m \cdot v = Q(M_\lambda)^m Q \cdot v.$$

From the convergence property of the sequence $\{\langle N, v_\lambda^{m*D}\rangle\}_{m=0}^\infty$ in Theorem 3.2.9, it follows immediately that

$$\lim_{m\to\infty} v_\lambda^{m*D} = Q \lim_{m\to\infty} (M_\lambda)^m Q \cdot v = QPDP^{-1}Q \cdot v = \hat{v}.$$

By Lemma 3.2.8, PD is a row-inessential coalitional matrix. Together with Lemma 3.2.7 and the fact that $P^{-1}Q$ is invertible, we derive that $PDP^{-1}Q$ is also a row-inessential coalitional matrix, that is to say $\langle N, PDP^{-1}Q \cdot v\rangle$ is an inessential game. $\langle N, QPDP^{-1}Q \cdot v\rangle$ is just the dual game of this game. Hence, by the self-duality property of the inessential game, the limit game $\langle N, \hat{v}\rangle$ is inessential.   □

Remark 3.4.5. *Notice that the limit game $\langle N, \hat{v}\rangle$ of the sequence of repeated dual similar associated games merely depends on the game $\langle N, v\rangle$ as $\hat{v} = QPDP^{-1}Q \cdot v$. The two limit games $\langle N, \tilde{v}\rangle$ and $\langle N, \hat{v}\rangle$ inherit the commutative relationship between the two types of associated games in Diagram 3.4.1. And for any player $i \in N$, the limit worth $\hat{v}(\{i\})$ is just the inner product of the i-th row of $QPDP^{-1}Q$ and the column vector v.*

3.5   Dual similar associated consistency for the Shapley value

In this section, we show that the Shapley value satisfies a new type consistency with respect to the dual similar associated game named dual similar associated consistency. Replacing the associated consistency in Hamiache's axiom system by the dual similar associated consistency, we axiomatize the Shapley value as the unique value satisfying the inessential game property, continuity and the dual similar associated consistency.

- (*Dual similar associated consistency*): a value Φ on $\mathcal{G}^N$ is called dual similar associated consistent, if for every game $\langle N, v \rangle$ and its dual similar associated game $\langle N, v_\lambda^D \rangle$, the value Φ satisfies $\Phi(v) = \Phi(v_\lambda^D)$.

The axiom of dual similar associated consistency means that any player receives the same payments in the original game and in the dual similar associated game. In matrix theory, the standard matrix M^{Sh} for the Shapley value is invariant under multiplication with the dual similar associated transformation matrix M_λ^D.

Lemma 3.5.1. *The Shapley value is consistent with respect to the dual similar associated game, that is $M^{Sh} = M^{Sh} M_\lambda^D$.*

Proof. Since $Sh(v) = M^{Sh}v$ and $Sh(v_\lambda^D) = M^{Sh}(M_\lambda^D \cdot v)$, it is sufficient to check the matrix equality $M^{Sh} M_\lambda^D = M^{Sh}$. By Lemma 3.4.2 and Proposition 2.3.5, we know $M_\lambda^D = Q M_\lambda Q$ and $M^{Sh} = M^{Sh} Q$. Together with Lemma 3.3.1, it follows that

$$M^{Sh} M_\lambda^D = M^{Sh} Q M_\lambda Q = M^{Sh} M_\lambda Q = M^{Sh} Q = M^{Sh}. \qquad \square$$

Theorem 3.5.2. *The Shapley value is the unique value satisfying the inessential game property, continuity and the dual similar associated consistency for $0 < \lambda < \frac{2}{n}$.*

Proof. Obviously, the Shapley value satisfies the inessential game property and continuity, and by Lemma 3.5.1 we know that the Shapley value verifies the dual similar associated consistency.

We concentrate only on the unicity proof. Consider a value Φ satisfying three listed properties. For any game $\langle N, v \rangle$, by both the dual similar associated consistency and continuity, it holds that $\Phi(v) = \Phi(\hat{v})$, where $\hat{v}$ is the limit game of the sequence of repeated dual similar associated games with $0 < \lambda < \frac{2}{n}$. By Lemma 3.2.9, $\langle N, \hat{v} \rangle$ is an inessential game. The inessential game property for Φ yields that $\Phi_i(N, \hat{v}) = \hat{v}(\{i\})$ for all $i \in N$. In summary, $\Phi(N, v) = (\hat{v}(\{i\}))_{i \in N}$.

From the proof of Theorem 3.3.2, we have $Sh(v) = Sh(\tilde{v})$, *i.e.*, $M^{Sh} = M^{Sh} P D P^{-1}$. Together with Proposition 2.3.5, $M^{Sh} = M^{Sh} Q$. It follows that

$$M^{Sh} = M^{Sh} P D P^{-1} \iff M^{Sh} Q = M^{Sh} P D P^{-1} Q$$

$$\iff M^{Sh} = M^{Sh} Q P D P^{-1} Q.$$

That is $Sh(v) = Sh(\hat{v})$. Since the game $\langle N, \hat{v} \rangle$ is inessential, we conclude that $Sh(v) = Sh(\hat{v}) = (\hat{v}(\{i\}))_{i \in N}$. Hence, $\Phi(v) = Sh(v)$.   □

Remark 3.5.3. *According to the proof of* Lemma 3.5.1, *the dual similar associated consistency for the Shapley value has been derived from Hamiache's associated consistency. We conclude this chapter with the proof of the converse statement. From the duality of the Shapley value and similarity between two types of associated consistency, we obtain that our dual similar associated consistency $M^{Sh} = M^{Sh} M_\lambda^D$ yields Hamiache's associated consistency for the Shapley value as:*

$$M^{Sh} M_\lambda = M^{Sh} Q M_\lambda^D Q = M^{Sh} M_\lambda^D Q = M^{Sh} Q = M^{Sh}.$$

So, the two types of associated consistency are equivalent for the Shapley value.

We prefer the matrix approach to these axiomatizations of the Shapley value in terms of the two types of associated consistency, although most of the results can be restated in game-theoretic text. Most of all, for axiomatizing the Shapley value, the procedure of diagonalization of the coalitional matrices M_λ, M_λ^D is very helpful to show and interpret the most important but difficult part of the convergence property of the sequences of repeated (dual similar) associated games. Moreover, the inessential property of the limit matrices $PDP^{-1}, QPDP^{-1}Q$ implies the structure of the limit games. In summary, this chapter illustrates that the diagonalization procedure and the similarity property for matrices can be applied successfully to cooperative game theory.

Chapter 4

Two types of $\mathcal{B}$-associated consistency for linear, symmetric and efficient values

In this chapter, we extend to the class of linear, symmetric, and efficient values what we did for the Shapley value in Chapter 3. At first, two types of games are introduced, the $\mathcal{B}$-scaled game and the $\mathcal{B}$-associated game. They are both linear transformations with special structures. For each value in this enlarged class, one explicit interrelationship to the Shapley value is restated as a matrix equality involving the $\mathcal{B}$-scaling matrix. Correspondingly, each value in this class is characterized by three axioms: the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency. In view of this axiomatization, these axioms imply traditional properties like linearity, efficiency and symmetry. By the $\mathcal{B}$-scaling procedure on the dual similar associated game, the $\mathcal{B}$-dual similar associated game as well as the $\mathcal{B}$-dual similar associated consistency are introduced and studied for characterizing this class of values. By the concept of $\mathcal{B}$-scaled game, the inverse problem of any linear, symmetric and efficient value is studied in terms of the inverse problem of the Shapley value, which is solved by analyzing the Shapley standard matrix and the associated transformation matrix.

4.1 $\mathcal{B}$-scaled game and $\mathcal{B}$-associated game

In [30], Driessen extended Hamiache's axiomatization of the Shapley value
to the enlarged class of linear, symmetric, and efficient values, of which the
Shapley value is the most important representative. The family of least square
values [73] as well as the *solidarity value* [65] are members of this class. For each
value in this enlarged class, one explicit interrelationship to the Shapley value
is exploited in order to present a uniform approach to obtain axiomatizations
of such values with reference to a slightly adapted inessential game property,
continuity, and a similar associated consistency. Following the former matrix
analysis on Hamiache's axiomatization of the Shapley value, a similar algebraic
approach is applicable to study Driessen's work.

Throughout this monograph, denote by $\mathcal{B} = \{b_s^n \mid n \geq 2, s = 1, 2, \cdots, n\}$,
a collection of non-zero scaling constants with $b_n^n = 1$ for all $n \geq 2$. For any
game $\langle N, v \rangle$, Driessen [30] defined its $\mathcal{B}$-*scaled game* $\langle N, \mathcal{B}v \rangle$ as

$$(\mathcal{B}v)(S) = b_s^n \cdot v(S), \quad \text{for all } S \in \Omega. \tag{4.1.1}$$

This $\mathcal{B}$-scaling of a game concerns a check of credibility of the characteristic
function by an independent arbiter. The task of the arbiter is to perform a
scaling procedure by taking into account the sizes n and s of both the player
set N and the coalition S (but not the members of the coalition themselves).
That is, for every coalition $S \in \Omega$, its initial worth $v(S)$ will be scaled down
or up to $b_s^n \cdot v(S)$ by some scaling number b_s^n. For example, if $b_s^n = \frac{1}{s}$, then the
scaling procedure involves averaging the worth of any coalition, different from
the grand coalition N. In real situations, the reader may ask for the scaling
constants b_s^n being positive, whereas in game-theoretic research it can be any
real number. For future purposes, we do not consider the case that b_s^n to be
zero. By convention, $b_n^n = 1$ for the sake of efficiency invariance expressed by
the invariant worth of the grand coalition.

We can rewrite the $\mathcal{B}$-scaled version of a game in terms of the $\mathcal{B}$-scaling
matrix B as follows.

Definition 4.1.1. *For any game $\langle N, v \rangle$ and any collection of scaling constants
$\mathcal{B}$, the $\mathcal{B}$-scaling game $\langle N, \mathcal{B}v \rangle$ is given by*

$$\mathcal{B}v = B \cdot v,$$

where the diagonal square-coalitional matrix $B = diag(b^n_{|S|})_{S \in \Omega}$ is called $\mathcal{B}$-scaling matrix.

Incorporating this scaling procedure, the $\mathcal{B}$-associated game $\langle N, v^{\mathcal{B}}_\lambda \rangle$ is meant to represent the optimistic self-evaluation of worths of coalitions, mathematically expressed as the associated game of the scaled version of the initial game.

Given any game $\langle N, v \rangle$, any collection $\mathcal{B}$ and $\lambda \in \mathbb{R}$, Driessen defined its $\mathcal{B}$-associated game $\langle N, v^{\mathcal{B}}_\lambda \rangle$ in [30] as follows:

$$(\mathcal{B}(v^{\mathcal{B}}_\lambda))(S) = (\mathcal{B}v)(S) + \lambda \sum_{j \in N \setminus S} \left[(\mathcal{B}v)(S \cup \{j\}) - (\mathcal{B}v)(S) - (\mathcal{B}v)(\{j\}) \right],$$

i.e., $v^{\mathcal{B}}_\lambda(S) = v(S) + \lambda \sum_{j \in N \setminus S} \left[\frac{b^n_{s+1}}{b^n_s} v(S \cup \{j\}) - v(S) - \frac{b^n_1}{b^n_s} v(\{j\}) \right]$, for all $S \in \Omega$.

That is,

$$v^{\mathcal{B}}_\lambda(S) = [1 - (n-s)\lambda]v(S) + \frac{\lambda b^n_{s+1}}{b^n_s} \sum_{j \in N \setminus S} v(S \cup \{j\}) - \frac{\lambda b^n_1}{b^n_s} \sum_{j \in N \setminus S} v(\{j\}).$$

Proposition 4.1.2. *For any game $\langle N, v \rangle$, any collection $\mathcal{B}$ and $\lambda \in \mathbb{R}$, it holds that $\langle N, \mathcal{B}(v^{\mathcal{B}}_\lambda) \rangle = \langle N, (\mathcal{B}v)^{Sh}_\lambda \rangle$. That is to say, the $\mathcal{B}$-scaled game $\langle N, \mathcal{B}(v^{\mathcal{B}}_\lambda) \rangle$ of the $\mathcal{B}$-associated game $\langle N, v^{\mathcal{B}}_\lambda \rangle$ is the associated game $\langle N, (\mathcal{B}v)^{Sh}_\lambda \rangle$ of the $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$.*

Analogously to the matrix approach for the associated game, we restate the $\mathcal{B}$-associated game as follows.

Definition 4.1.3. *For any game $\langle N, v \rangle$, any collection $\mathcal{B}$ and $\lambda \in \mathbb{R}$, the $\mathcal{B}$-associated game $\langle N, v^{\mathcal{B}}_\lambda \rangle$ is given by*

$$v^{\mathcal{B}}_\lambda = M^{\mathcal{B}}_\lambda \cdot v,$$

where the matrix $M^{\mathcal{B}}_\lambda$ is square-coalitional given by $[M^{\mathcal{B}}_\lambda]_{S,T} = \frac{b^n_t}{b^n_s}[M_\lambda]_{S,T}$ for all $S, T \in \Omega$. And the sequence of repeated $\mathcal{B}$-associated games $\{\langle N, v^{m*\mathcal{B}}_\lambda \rangle\}^\infty_{m=0}$ is defined recursively as

$$v^{m*\mathcal{B}}_\lambda = M^{\mathcal{B}}_\lambda \cdot v^{(m-1)*\mathcal{B}}_\lambda, \quad \text{for all } m \geq 1, \quad \text{where } v^{0*\mathcal{B}}_\lambda = v.$$

We write M_λ instead of $M_\lambda^{\mathcal{B}}$ if it concerns the unitary constants $b_s^n = 1$ for all $1 \leq s \leq n$, and then $\langle N, v_\lambda^{\mathcal{B}} \rangle$ agrees with the associated game $\langle N, v_\lambda^{Sh} \rangle$. Next we show that the $\mathcal{B}$-associated transformation matrix $M_\lambda^{\mathcal{B}}$ inherits certain properties from the associated transformation matrix M_λ.

Proposition 4.1.4. *Let M_λ and $M_\lambda^{\mathcal{B}}$ be the associated transformation matrix and the $\mathcal{B}$-associated transformation matrix, respectively.*

1. *$M_\lambda^{\mathcal{B}} = B^{-1} M_\lambda B$, where $B = diag(b_{|S|}^n)_{S \in \Omega}$.*

2. *$M_\lambda^{\mathcal{B}}$ and M_λ have the same eigenvalues and the same (algebraic) multiplicities of eigenvalues. And $\vec{y}$ is an eigenvector of $M_\lambda^{\mathcal{B}}$ if and only if $B\vec{y}$ is an eigenvector of M_λ.*

3. *If $0 < \lambda < \frac{2}{n}$, then $\lim_{m \to \infty} (M_\lambda^{\mathcal{B}})^m = B^{-1} \lim_{m \to \infty} (M_\lambda)^m B = B^{-1} P D P^{-1} B$.*

Proof. 1. Since B is a diagonal matrix such that $b_s^n \neq 0$ for all $1 \leq s \leq n$, its inverse $B^{-1} = diag(\frac{1}{b_{|S|}^n})_{S \in \Omega}$ is also a diagonal matrix. For any $S, T \in \Omega$, we have

$$[B^{-1} M_\lambda B]_{S,T} = \sum_{R \in \Omega} [B^{-1}]_{S,R} [M_\lambda B]_{R,T} = [B^{-1}]_{S,S} [M_\lambda B]_{S,T}$$

$$= \frac{1}{b_s^n} \sum_{R \in \Omega} [M_\lambda]_{S,R} [B]_{R,T} = \frac{1}{b_s^n} [M_\lambda]_{S,T} [B]_{T,T}$$

$$= \frac{b_t^n}{b_s^n} [M_\lambda]_{S,T} = [M_\lambda^{\mathcal{B}}]_{S,T}.$$

Thus, the similarity property $M_\lambda^{\mathcal{B}} = B^{-1} M_\lambda B$ holds.

2. By the latter similarity property, it is known that $M_\lambda^{\mathcal{B}}$ and M_λ have the same eigenvalues and the same multiplicities of eigenvalues. Let $\vec{y}$ be an eigenvector of $M_\lambda^{\mathcal{B}}$ corresponding to eigenvalue μ. Then

$$M_\lambda^{\mathcal{B}} \vec{y} = \mu \vec{y} \iff (B^{-1} M_\lambda B)\vec{y} = \mu \vec{y} \iff M_\lambda(B\vec{y}) = B(\mu \vec{y}) = \mu(B\vec{y}).$$

Clearly, μ is an eigenvalue of M_λ and $B\vec{y}$ is a corresponding eigenvector of M_λ.

3. It is derived immediately from conclusion 1 and Lemma 3.2.7. $\square$

Concerning any linear, symmetric and efficient value on $\mathcal{G}^N$, Ruiz *et al.* ([73], Lemma 9, page 117) characterized its value payoff vector $\Phi(v)$ for every

game $\langle N, v \rangle$ to be of the following form

$$\Phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N \\ S \ni i}} \frac{\rho_s^n}{s} \cdot v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} \frac{\rho_s^n}{n-s} \cdot v(S), \quad \text{for all } i \in N, \quad (4.1.2)$$

for a certain sequence of real numbers ρ_s^n, $s = 1, 2, \cdots, n-1$ (as presented in Corollary 2.2.8). Two equivalent formulae were presented by Driessen ([30], Equivalence Theorem 1.4). Note that the Shapley value is widely studied because of its many interesting properties and axiomatic characterizations as well. For our future purposes, we interpret Ruiz' formula as the Shapley value of an appropriately chosen $\mathcal{B}$-scaled game as follows. Its algebraic formulation is stated in the subsequent corollary.

Theorem 4.1.5 (Driessen [30]). *A value Φ on $\mathcal{G}^N$ satisfies linearity, symmetry and efficiency if and only if there exists a collection of constants $\mathcal{B}$ such that, for all games $\langle N, v \rangle$, the value $\Phi(v) = Sh(\mathcal{B}v)$.*

Corollary 4.1.6. *A value Φ on $\mathcal{G}^N$ satisfies linearity, symmetry and efficiency if and only if there exists a $\mathcal{B}$-scaling matrix $B = diag(b_{|S|}^n)_{S \in \Omega}$ such that, for all games $\langle N, v \rangle$, the value $\Phi(v) = M^{Sh}Bv$.*

Recalling the alternative formula of the Shapley value in Corollary 2.3.4, we obtain directly the following alternative formula of linear, symmetric and efficient values.

Corollary 4.1.7. *A value Φ on $\mathcal{G}^N$ satisfies linearity, symmetry and efficiency if and only if there exists a sequence of real numbers b_s^n, $s = 1, 2, \cdots, n-1$ such that, for all games $\langle N, v \rangle$, all $i \in N$,*

$$\Phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} \frac{(s-1)!(n-s)!}{n!} \Big[b_s^n v(S) - b_{n-s}^n v(N \setminus S) \Big].$$

We should mention that an equivalent formula has been obtained by Lamoneda, Garcia and Sanchez [43] by using representation theory and another proof can be found in their paper [44].

In the setting of these results and formulae, the constants ρ_s^n and b_s^n, $s = 1, 2, \cdots, n-1$ are arbitrary real numbers. In fact, for the $\mathcal{B}$-scaling version of a game, these scaling constants in $\mathcal{B} = \{ b_s^n \mid n \geq 2, s = 1, 2, \cdots, n \}$ can be chosen as any real number. We prefer non-zero constants for the $\mathcal{B}$-associated game to be well-defined.

4.2 $\mathcal{B}$-associated consistency

In this section, we aim to characterize each linear, symmetric and efficient value in terms of an adaption of associated consistency. It is based on the representation for such values by the matrix approach in terms of the Shapley value. We introduce the $\mathcal{B}$-associated consistency for a value on $\mathcal{G}^N$, which generalizes the consistency with respect to the associated game.

- $\mathcal{B}$-*associated consistency*: a value Φ on $\mathcal{G}^N$ possesses $\mathcal{B}$-associated consistency with respect to the $\mathcal{B}$-associated game if $\Phi(v_\lambda^{\mathcal{B}}) = \Phi(v)$ for all games $\langle N, v \rangle$ and all $\lambda \in \mathbb{R}$.

According to the next theorem, the $\mathcal{B}$-associated game is chosen in such a way as to guarantee that the corresponding linear, symmetric and efficient value Φ satisfies the $\mathcal{B}$-associated consistency.

Theorem 4.2.1 (Driessen [30]). *For a given collection of constants $\mathcal{B}$, let Φ be the linear, symmetric and efficient value on $\mathcal{G}^N$ with the representation matrix $M^\Phi = M^{Sh}B$. Then Φ satisfies the $\mathcal{B}$-associated consistency.*

Proof by the matrix approach. In view of Corollary 4.1.6, we show that

$$M^{Sh}B \cdot v_\lambda^{\mathcal{B}} = M^{Sh}B \cdot v \quad \text{for all games } \langle N, v \rangle, \text{ and all } \lambda \in \mathbb{R}.$$

Since $v_\lambda^{\mathcal{B}} = M_\lambda^{\mathcal{B}} \cdot v$, it is sufficient to check $M^{Sh}BM_\lambda^{\mathcal{B}} = M^{Sh}B$. From Proposition 4.1.4 (1) and Lemma 3.3.1, we conclude that

$$M^{Sh}BM_\lambda^{\mathcal{B}} = M^{Sh}BB^{-1}M_\lambda B = M^{Sh}M_\lambda B = M^{Sh}B. \qquad \square$$

Definition 4.2.2 (Driessen [30]). *A value Φ on $\mathcal{G}^N$ verifies the $\mathcal{B}$-inessential game property with respect to a given collection of constants $\mathcal{B}$ if the value satisfies $\Phi_i(v) = b_1^n \cdot v(\{i\})$, for all $\mathcal{B}$-inessential games $\langle N, v \rangle$, and all $i \in N$. Here the game $\langle N, v \rangle$ is called $\mathcal{B}$-inessential if the $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$ is inessential.*

Similar to the result in Theorem 3.2.9 about the convergence of the sequence of repeated associated games, the next theorem states the convergence of the sequence of repeated $\mathcal{B}$-associated games.

Theorem 4.2.3. *Let $0 < \lambda < \frac{2}{n}$. The sequence of repeated $\mathcal{B}$-associated games $\{\langle N, v_\lambda^{m*\mathcal{B}} \rangle\}_{m=0}^\infty$ converges to the game $\langle N, \bar{v} \rangle$, where $\bar{v} = B^{-1}PDP^{-1}B \cdot v$. Furthermore, the limit game $\langle N, \bar{v} \rangle$ is $\mathcal{B}$-inessential.*

Proof. By Proposition 4.1.4 (3), the sequence of games $\{\langle N, v_\lambda^{m*\mathcal{B}} \rangle\}_{m=0}^\infty$ converges to

$$\bar{v} = \lim_{m \to \infty} (M_\lambda^{\mathcal{B}})^m \cdot v = B^{-1}PDP^{-1}B \cdot v.$$

So, $\mathcal{B}\bar{v} = B \cdot \bar{v} = PDP^{-1}B \cdot v$. By Lemma 3.2.8 (3), the matrix PD is row-inessential, and it follows from Lemma 3.2.7 (2) that the matrix $PDP^{-1}B$ is row-inessential too. Hence, by Lemma 3.2.7 (3), the game $\langle N, \mathcal{B}\bar{v} \rangle$ is inessential, *i.e.*, the limit game $\langle N, \bar{v} \rangle$ is $\mathcal{B}$-inessential. □

Remark 4.2.4. *The limit game $\langle N, \bar{v} \rangle$ of the sequence of repeated $\mathcal{B}$-associated games merely depends on the game $\langle N, v \rangle$ as $\bar{v} = B^{-1}PDP^{-1}B \cdot v$. The two limit games $\langle N, \bar{v} \rangle$ and $\langle N, \tilde{v} \rangle$ inherit the commutative relationship in* Proposition 4.1.2.

So far, we have presented three properties of a value on the game space $\mathcal{G}^N$, which are the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency. In the following we show that any linear, symmetric and efficient value possesses these three properties.

Lemma 4.2.5 (Driessen [30]). *For a given collection of constants $\mathcal{B}$ and any game $\langle N, v \rangle$, the corresponding linear, symmetric and efficient value $\Phi(v) = Sh(\mathcal{B}v) = M^{Sh}B \cdot v$ satisfies the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency.*

Proof by the matrix approach. By Theorem 4.2.1, the value Φ satisfies the $\mathcal{B}$-associated consistency. If the $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$ is inessential, then

$$\Phi_i(v) = Sh_i(\mathcal{B}v) = (\mathcal{B}v)(\{i\}) = b_1^n \cdot v(\{i\}), \quad \text{for all } i \in N.$$

So Φ verifies the $\mathcal{B}$-inessential game property. Let us consider any convergent sequence of games $\{\langle N, v_k \rangle\}_{k=0}^\infty$, say the limit of which is the game $\langle N, \bar{v} \rangle$. The corresponding sequence of values $\{\Phi(v_k)\}_{k=0}^\infty$ is the sequence $\{M^{Sh}B \cdot v_k\}_{k=0}^\infty$. It converges to $\Phi(\bar{v})$ of the limit game $\langle N, \bar{v} \rangle$. This shows the continuity of the value Φ. □

We present an alternative, algebraic proof for Driessen's axiomatization of any linear, symmetric and efficient value.

Theorem 4.2.6 (Driessen [30]). *For a given collection of constants $\mathcal{B}$, there exists a unique value Φ on $\mathcal{G}^N$ satisfying the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency with $0 < \lambda < \frac{2}{n}$, and the value Φ is the linear, symmetric, and efficient value induced by $\mathcal{B}$, i.e., $\Phi(v) = Sh(\mathcal{B}v)$ for all games $\langle N, v \rangle$.*

Proof by the matrix approach. By Lemma 4.2.5, we only concentrate on the uniqueness proof. Consider a value Φ satisfying the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency with $0 < \lambda < \frac{2}{n}$. For any game $\langle N, v \rangle$, we show that $\Phi(v) = Sh(\mathcal{B}v)$. By both the $\mathcal{B}$-associated consistency and continuity, it holds that

$$\Phi(v) = \Phi(\bar{v}), \quad \text{where } \bar{v} = B^{-1}PDP^{-1}B \cdot v.$$

By Theorem 4.2.3, the limit game $\langle N, \bar{v} \rangle$ is $\mathcal{B}$-inessential. So, the $\mathcal{B}$-inessential game property for Φ yields that $\Phi_i(\bar{v}) = b_1^n \cdot \bar{v}(\{i\})$ for all $i \in N$. In summary,

$$\Phi(v) = b_1^n \cdot (\bar{v}(\{i\}))_{i \in N}.$$

By the proof of Theorem 3.3.2, it holds that $M^{Sh} = M^{Sh}PDP^{-1}$. We conclude that

$$M^{Sh}B \cdot v = M^{Sh}PDP^{-1}B \cdot v = M^{Sh}BB^{-1}PDP^{-1}B \cdot v = M^{Sh}B \cdot \bar{v}.$$

That is, $Sh(\mathcal{B}v) = Sh(\mathcal{B}\bar{v})$. Since the game $\langle N, \mathcal{B}\bar{v} \rangle$ is inessential and the Shapley value possess the inessential game property, it follows that

$$Sh(\mathcal{B}v) = Sh(\mathcal{B}\bar{v}) = b_1^n \cdot (\bar{v}(\{i\}))_{i \in N}.$$

Hence, $\Phi(v) = Sh(\mathcal{B}v)$.   □

4.3 $\mathcal{B}$-dual similar associated consistency

Following the technique to construct linear operators on the game space, we present a new associated game by considering both $\mathcal{B}$-scaling and dual operator for studying the class of linear, symmetric and efficient values.

As we showed in Proposition 4.1.2, the $\mathcal{B}$-scaled game of the $\mathcal{B}$-associated game is the associated game of the $\mathcal{B}$-scaled game or, in matrix representation

$$M_\lambda^{\mathcal{B}} = B^{-1} M_\lambda B.$$

Replacing the associated game by the dual-associated game or the associated transformation matrix M_λ by the dual similar associated transformation matrix M_λ^D, we introduce the $\mathcal{B}$-dual similar associated transformation matrix $M_\lambda^{\mathcal{B}D}$ and the corresponding $\mathcal{B}$-dual similar associated game as follows.

Definition 4.3.1. *For any collection $\mathcal{B}$ and $\lambda \in \mathbb{R}$, the $\mathcal{B}$-dual similar associated transformation matrix $M_\lambda^{\mathcal{B}D}$ is defined by*

$$M_\lambda^{\mathcal{B}D} = B^{-1} M_\lambda^D B, \quad \text{or equivalently,} \quad M_\lambda^{\mathcal{B}D} = B^{-1} Q M_\lambda Q B,$$

where B and Q are the $\mathcal{B}$-scaling matrix and the dual matrix, respectively.

For any game $\langle N, v \rangle$, the $\mathcal{B}$-dual similar associated game $\langle N, v_\lambda^{\mathcal{B}D} \rangle$ is defined by

$$v_\lambda^{\mathcal{B}D} = M_\lambda^{\mathcal{B}D} \cdot v.$$

The sequence of repeated $\mathcal{B}$-dual similar associated games $\{\langle N, v_\lambda^{m*\mathcal{B}D} \rangle\}_{m=0}^{\infty}$ is defined recursively as

$$v_\lambda^{m*\mathcal{B}D} = M_\lambda^{\mathcal{B}D} \cdot v_\lambda^{(m-1)*\mathcal{B}D}, \quad \text{for all } m \geq 1, \quad \text{where } v_\lambda^{0*\mathcal{B}D} = v.$$

In the characteristic function form, the $\mathcal{B}$-dual similar associated game $\langle N, v_\lambda^{\mathcal{B}D} \rangle$ of any game $\langle N, v \rangle$ and all $\lambda \in \mathbb{R}$ is given by, for all $S \in \Omega$,

$$v_\lambda^{\mathcal{B}D}(S) = v(S) + \lambda \sum_{j \in S} \left\{ \left[\frac{b_n^n}{b_s^n} v(N) - \frac{b_{n-1}^n}{b_s^n} v(N \backslash \{j\}) \right] - \left[v(S) - \frac{b_{s-1}^n}{b_s^n} v(S \backslash \{j\}) \right] \right\}.$$

Its worth $v_\lambda^{\mathcal{B}D}(S)$ differs from the initial worth $v(S)$ by taking into account the weighted fractions of net benefits for all members of S, as described by the right hand of the above formula. Notice that $v_\lambda^{\mathcal{B}D}(N) = v(N)$, which implies that the efficiency is inherited from the original game. For all $S \in \Omega$, we can express the worth $v_\lambda^{\mathcal{B}D}(S)$ as

$$v_\lambda^{\mathcal{B}D}(S) = (1 - s\lambda) v(S) + \frac{b_n^n}{b_s^n} s\lambda v(N) - \frac{b_{n-1}^n}{b_s^n} \lambda \sum_{j \in S} v(N \backslash \{j\}) + \frac{b_{s-1}^n}{b_s^n} \lambda \sum_{j \in S} v(S \backslash \{j\}).$$

From this formula we derive that the entries of the square-coalitional matrix $M_\lambda^{\mathcal{BD}} = \left[M_\lambda^{\mathcal{BD}}\right]_{S,T \in \Omega}$ are as follows.

$$
\left[M_\lambda^{\mathcal{BD}}\right]_{S,T} = \begin{cases} 1 - s\lambda, & \text{if } T = S \text{ and } S \neq N; \\[2mm] \frac{b_n^n}{b_s^n} s\lambda, & \text{if } T = N \text{ and } S \neq N; \\[2mm] \frac{b_{s-1}^n}{b_s^n} \lambda, & \text{if } T = S \setminus \{j\}, j \in S \text{ and } S \neq N; \\[2mm] -\frac{b_{n-1}^n}{b_s^n} \lambda, & \text{if } T = N \setminus \{j\}, j \in S \text{ and } S \neq N; \\[2mm] 1, & \text{if } T = S = N; \\[2mm] 0, & \text{otherwise.} \end{cases}
$$

The following proposition is derived directly from Definition 4.3.1 and Proposition 4.1.4.

Proposition 4.3.2. *If* $0 < \lambda < \frac{2}{n}$, *then*

$$
\lim_{m \to \infty} (M_\lambda^{\mathcal{BD}})^m = B^{-1} Q \lim_{m \to \infty} (M_\lambda)^m QB = B^{-1} QPDP^{-1} QB.
$$

This implies the convergence of the sequence of repeated $\mathcal{B}$-dual similar associated games, which is similar to the sequence of repeated $\mathcal{B}$-associated games in Theorem 4.2.3.

Theorem 4.3.3. *Let* $0 < \lambda < \frac{2}{n}$. *The sequence of repeated* $\mathcal{B}$-*dual similar associated games* $\{\langle N, v_\lambda^{m*\mathcal{BD}} \rangle\}_{m=0}^\infty$ *converges to the game* $\langle N, \ddot{v} \rangle$, *where* $\ddot{v} = B^{-1} QPDP^{-1} QB \cdot v$. *Furthermore, the limit game* $\langle N, \ddot{v} \rangle$ *is* $\mathcal{B}$-*inessential.*

Proof. It is derived directly from Proposition 4.3.2 that the sequence of games $\{\langle N, v_\lambda^{m*\mathcal{BD}} \rangle\}_{m=0}^\infty$ converges to

$$
\ddot{v} = \lim_{m \to \infty} (M_\lambda^{\mathcal{BD}})^m \cdot v = B^{-1} QPDP^{-1} QB \cdot v.
$$

So, $\mathcal{B}\ddot{v} = B \cdot \ddot{v} = QPDP^{-1} QB \cdot v$. By Lemma 3.2.8 (3), the matrix PD is row-inessential, and it follows from Lemma 3.2.7 (2) that the matrix $PDP^{-1} QB$ is row-inessential too. That is to say that $\langle N, PDP^{-1} QB \cdot v \rangle$ is an inessential game. By the self-duality property of the inessential game, $\langle N, QPDP^{-1} QB \cdot v \rangle$ is inessential, *i.e.*, the game $\langle N, \mathcal{B}\ddot{v} \rangle$ is inessential. Hence, the limit game $\langle N, \ddot{v} \rangle$ is $\mathcal{B}$-inessential. $\square$

With respect to the $\mathcal{B}$-dual similar associated game, we define a new type of associated consistency, named $\mathcal{B}$-dual similar associated consistency, as follows.

- $\mathcal{B}$-*dual similar associated consistency*: a value Φ on $\mathcal{G}^N$ possesses $\mathcal{B}$-dual similar associated consistency with respect to the $\mathcal{B}$-dual similar associated game if $\Phi(v_\lambda^{\mathcal{B}D}) = \Phi(v)$, for all games $\langle N, v \rangle$ and all $\lambda \in \mathbb{R}$.

The next theorem shows that a $\mathcal{B}$-associated game can be chosen for guaranteeing that the corresponding linear, symmetric and efficient value Φ verifies $\mathcal{B}$-dual similar associated consistency.

Theorem 4.3.4. *For a given collection of constants $\mathcal{B}$, let Φ be the linear, symmetric and efficient value on $\mathcal{G}^N$ with the representation matrix $M^\Phi = M^{Sh}B$. Then Φ satisfies the $\mathcal{B}$-dual similar associated consistency.*

Proof. In view of Corollary 4.1.6, it is sufficient to show that $M^{Sh}B \cdot v_\lambda^{\mathcal{B}D} = M^{Sh}B \cdot v$, for all games $\langle N, v \rangle$ and all $\lambda \in \mathbb{R}$. By Definition 4.3.1, Proposition 2.3.5 and Lemma 3.3.1, we conclude that

$$M^{Sh}BM_\lambda^{\mathcal{B}D} = M^{Sh}B \cdot B^{-1}QM_\lambda QB = M^{Sh}QM_\lambda QB$$

$$= M^{Sh}M_\lambda QB = M^{Sh}QB = M^{Sh}B. \qquad \square$$

Replacing the $\mathcal{B}$-associated consistency of Driessen's axiomatization, we characterize any linear, symmetric and efficient value in terms of the $\mathcal{B}$-dual associated consistency, together with the $\mathcal{B}$-inessential game property and continuity.

Theorem 4.3.5. *For a given collection of constants $\mathcal{B}$, there exists a unique value Φ on $\mathcal{G}^N$ satisfying the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-dual similar associated consistency with $0 < \lambda < \frac{2}{n}$, and the value Φ is the linear, symmetric, and efficient value induced by $\mathcal{B}$, i.e., $\Phi(v) = Sh(\mathcal{B}v)$ for all games $\langle N, v \rangle$.*

Proof. Let Φ be a linear, symmetric, and efficient value with the representation matrix $M^\Phi = M^{Sh}B$. By Theorem 4.3.4, Φ verifies the $\mathcal{B}$-dual similar associated consistency. Similar to the proof of Lemma 4.2.5, we can show that Φ is continuous and satisfies the $\mathcal{B}$-inessential game property.

Let us turn to the unicity proof now. Consider a value Φ having these three properties with $0 < \lambda < \frac{2}{n}$. For any game $\langle N, v \rangle \in \mathcal{G}^N$, by both the $\mathcal{B}$-dual similar associated consistency and continuity, we have

$$\Phi(v) = \Phi(\ddot{v}), \quad \text{where } \ddot{v} = B^{-1}QPDP^{-1}QB \cdot v.$$

By Theorem 4.3.3, the limit game $\langle N, \ddot{v} \rangle$ is $\mathcal{B}$-inessential. We conclude that $\Phi(v) = b_1^n \cdot (\ddot{v}(\{i\}))_{i \in N}$.

The proof of Theorem 3.5.2 yields that $M^{Sh} = M^{Sh}QPDP^{-1}Q$. It follows that

$$M^{Sh}B \cdot v = M^{Sh}QPDP^{-1}QB \cdot v = M^{Sh}BB^{-1}QPDP^{-1}QB \cdot v = M^{Sh}B \cdot \ddot{v}.$$

That is to say, $Sh(\mathcal{B}v) = Sh(\mathcal{B}\ddot{v})$. Since the game $\langle N, \mathcal{B}\ddot{v} \rangle$ is inessential and the Shapley value satisfies the inessential game property, we conclude that

$$Sh(\mathcal{B}v) = Sh(\mathcal{B}\ddot{v}) = b_1^n \cdot (\ddot{v}(\{i\}))_{i \in N}.$$

Hence, $\Phi(v) = Sh(\mathcal{B}v)$. $\square$

Remark 4.3.6. *Similar to* Remark 3.5.3, *for a given collection of constants* $\mathcal{B}$, *we remark that the* $\mathcal{B}$-*associated consistency and the* $\mathcal{B}$-*dual similar associated consistency are equivalent for the linear, symmetric and efficient value induced by* $\mathcal{B}$.

4.4 The inverse problem

The equivalence for non-transferable utility (NTU) games has been studied in the work of Aumann and Kurz [4], Aumann, Kurz and Neyman [5]. In each of these papers, strikingly different games were found to have the same value. Formally, for a value Φ on $\mathcal{G}^N$, two games $\langle N, v \rangle, \langle N, w \rangle$ are called equivalent if $\Phi(v) = \Phi(w)$. Any linear value Φ on $\mathcal{G}^N$ can be viewed as a linear operator Φ on the vector space $\mathbb{R}^{2^n - 1}$. The following problem is often considered. Given any vector $b \in \mathbb{R}^N$, find the set of games $\langle N, v \rangle$ such that $\Phi(v) = b$. We call it the *inverse problem* of the value Φ. The null space $\mathcal{N}_\Phi$ of Φ is defined as the subspace of these games $v \in \mathbb{R}^{2^n - 1}$ such that $\Phi(v) = \vec{0}$. Clearly, the inverse problem of a linear value Φ is to solve the null space $\mathcal{N}_\Phi$. To demonstrate the

equivalence of two games $v, w \in \mathbb{R}^{2^n-1}$, we need to show that the difference game $v - w$ is in the null space $\mathcal{N}_\Phi$ of the linear value Φ. On the other hand, every game can be decomposed in a unique manner as the sum of its value game (a game with the same value) and an game of $\mathcal{N}_\Phi$. We thus see that, quite apart from what one might think, inessential games, as one type of value games, play a significant role in the characterization of the Shapley value.

Using the representation theory of symmetric groups, Kleinberg and Weiss [52] constructed a direct-sum decomposition of the null space of the Shapley value into invariant subspaces. Then they used the same theory to derive a characterization of a very general type of value, of which the Shapley value is one particular example. The inverse problem of the Shapley value was also studied by Dragan [20]. He presented a potential basis for the null space and an explicit representation of all games with an apriori given Shapley value. The same potential approach was used to analyze the null space of the Banzhaf value (Dragan [22]), as well the family of semivalues (Dragan [23]).

In terms of the Shapley standard matrix M^{Sh}, recall the matrix representation $Sh(v) = M^{Sh} \cdot v$ for all $v \in \mathbb{R}^{2^n-1}$. Therefore, the null space $\mathcal{N}_{Sh}$ of the Shapley value agrees with the null space of the matrix M^{Sh}, *i.e.*,

$$\mathcal{N}_{Sh} = \text{Null}(M^{Sh}) = \{x \in \mathbb{R}^{2^n-1} \mid M^{Sh}x = \vec{0}\}.$$

For a matrix A, the dimension of its null space is denoted by $\dim\text{Null}(A)$. The well-known Rank Theorem in algebra theory is as follows.

Theorem 4.4.1 (The Rank Theorem [54]). *If A is an $n \times m$ matrix, then*

$$rank(A) + dimNull(A) = m.$$

Inspired by the associated consistency, the inverse problem of the Shapley value is studied in terms of the associated transformation matrix M_λ.

Theorem 4.4.2. *The null space of M^{Sh} is the column space of $M_\lambda - I$, i.e.,*

$$Null(M^{Sh}) = Col(M_\lambda - I).$$

Proof. By the associated consistency of the Shapley value in Lemma 3.3.1, we have $M^{Sh}M_\lambda = M^{Sh}$, or equivalently, $M^{Sh}(M_\lambda - I) = \mathbf{0}$. Hence,

$$\text{Col}(M_\lambda - I) \subseteq \text{Null}(M^{Sh}).$$

It is sufficient to show that these two spaces have the same dimension. By Proposition 3.2.3, 1 is an eigenvalue of M_λ and $\mathrm{rank}(M_\lambda - I) = 2^n - 1 - n$. Therefore, the dimension of the column space $\mathrm{Col}(M_\lambda - I)$ is

$$\mathrm{dimCol}(M_\lambda - I) = \mathrm{rank}(M_\lambda - I) = 2^n - 1 - n.$$

By the Rank Theorem and Proposition 2.3.6,

$$\mathrm{dimNull}(M^{Sh}) = 2^n - 1 - \mathrm{rank}(M^{Sh}) = 2^n - 1 - n.$$

Therefore, $\mathrm{Null}(M^{Sh}) = \mathrm{Col}(M_\lambda - I)$. $\square$

Remark 4.4.3. *Since $M_\lambda = PD_\lambda P^{-1}$ in Lemma 3.2.8, it follows that*

$$M_\lambda - I = P(D_\lambda - I)P^{-1},$$

and the columns in the diagonal matrix $D_\lambda - I$ corresponding to the eigenvalue 1, which are indexed by all single player coalitions, is a zero-vector. The other columns span the column space $\mathrm{Col}(M_\lambda - I)$, equivalently, they are a basis for the null space $\mathcal{N}_{Sh}$.

For any linear, symmetric and efficient value Φ with respect to a collection of constants $\mathcal{B}$, its inverse problem or the null space $\mathcal{N}_\Phi$ can be discussed in terms of the $\mathcal{B}$-associated consistency. Here, $\mathcal{N}_\Phi$ is derived directly from the relationship between Φ and the Shapley value as given in Corollary 4.1.6.

Corollary 4.4.4. *Let Φ be a linear, symmetric and efficient value with respect to a collection of constants $\mathcal{B}$. Then the $\mathcal{B}$-scaling of the null space $\mathcal{N}_\Phi$ of Φ is the null space $\mathcal{N}_{Sh}$ of the Shapley value, i.e.,*

$$\mathcal{N}_\Phi = \{v \mid \mathcal{B}v \in \mathcal{N}_{Sh}\}.$$

Since the collection of constants $\mathcal{B}$ are non-zero, so B is invertible and the null space $\mathcal{N}_\Phi$ of Φ is the linear transformation by B^{-1} on the null space $\mathcal{N}_{Sh}$, i.e., $\mathcal{N}_\Phi = \{B^{-1} \cdot w \mid w \in \mathcal{N}_{Sh}\}$.

4.5   Conclusions about matrix analysis

Chapters 2 to 4 deal with the class of linear, symmetric and efficient values, of which the Shapley value is the most important representative. For any linear value being identified by a corresponding coalitional matrix, the algebraic

representation and the matrix analysis are used to characterize these values. Each value in this class is characterized by three axioms: the $\mathcal{B}$-inessential game property, continuity, and the $\mathcal{B}$-associated consistency (respectively, the $\mathcal{B}$-dual similar associated consistency). As a by-product we use neither the linearity nor the efficiency axiom. For this axiomatization, several types of games are introduced on the game space. They are linear transformations with special structures, which are associated with some types of consistency for the class of values. Concerning the matrix approach for the $\mathcal{B}$-associated consistency (respectively, the $\mathcal{B}$-dual similar associated consistency) of such values, especially the associated consistency (respectively, the dual similar associated consistency) of the Shapley value, the following three tables summarize the relevant matrices, games and their mutual relationships.

Matrix	Name of matrix	Value/Game	Definition
M^{Φ}	representation	$\Phi(v) = M^{\Phi} \cdot v$	Definition
M^{Sh}	Shapley standard	$Sh(v) = M^{Sh} \cdot v$	Def. 2.3.2
Q	dual	$v^{*} = Q \cdot v$	Def. 2.1.3
B	$\mathcal{B}$-scaling diagonal	$\mathcal{B}v = B \cdot v$	Def. 4.1.1
M_{λ}	associated transformation	$v_{\lambda}^{Sh} = M_{\lambda} \cdot v$	Def. 3.2.1
M_{λ}^{D}	dual similar associated transf.	$v_{\lambda}^{D} = M_{\lambda}^{D} \cdot v$	Def. 3.4.1
$M_{\lambda}^{\mathcal{B}}$	$\mathcal{B}$-associated transformation	$v_{\lambda}^{\mathcal{B}} = M_{\lambda}^{\mathcal{B}} \cdot v$	Def. 4.1.3
$M_{\lambda}^{\mathcal{B}D}$	$\mathcal{B}$-dual similar associated transf.	$v_{\lambda}^{\mathcal{B}D} = M_{\lambda}^{\mathcal{B}D} \cdot v$	Def. 4.3.1

Table 4.5.1: Matrix representation of Shapley value and games

Sequence	Limit game	Property	Result
$\{\langle N, v_{\lambda}^{m*Sh} \rangle\}_{m=0}^{\infty}$	$\tilde{v} = PDP^{-1} \cdot v$	inessential	Thm. 3.2.9
$\{\langle N, v_{\lambda}^{m*D} \rangle\}_{m=0}^{\infty}$	$\hat{v} = QPDP^{-1}Q \cdot v$	inessential	Thm. 3.4.4
$\{\langle N, v_{\lambda}^{m*\mathcal{B}} \rangle\}_{m=0}^{\infty}$	$\bar{v} = B^{-1}PDP^{-1}B \cdot v$	$\mathcal{B}$-inessential	Thm. 4.2.3
$\{\langle N, v_{\lambda}^{m*\mathcal{B}D} \rangle\}_{m=0}^{\infty}$	$\ddot{v} = B^{-1}QPDP^{-1}QB \cdot v$	$\mathcal{B}$-inessential	Thm. 4.3.3

Table 4.5.2: Convergence results

Property	Matrix equality	Result
duality	$M^{Sh} = M^{Sh}Q$	Prop. 2.3.5
similarity	$M_\lambda^D = QM_\lambda Q$	Lem. 3.4.2
	$M_\lambda = QM_\lambda^D Q$	Coro. 3.4.3
associated consistency	$M^{Sh} = M^{Sh}M_\lambda$	Lem. 3.3.1
dual similar associated consist.	$M^{Sh} = M^{Sh}M_\lambda^D$	Lem. 3.5.1
$\mathcal{B}$-associated consist.	$M^{Sh}B = M^{Sh}BM_\lambda^{\mathcal{B}}$	Thm. 4.2.1
$\mathcal{B}$-dual similar associated consist.	$M^{Sh}B = M^{Sh}BM_\lambda^{\mathcal{B}D}$	Thm. 4.3.4

Table 4.5.3: Associated consistency results

Chapter 5

Consistency for linear, symmetric and efficient values

This chapter contributes to consistency (*i.e.*, the reduced game property) in cooperative game theory. The most popular definitions of reduced games throughout the literature on cooperative game theory are reviewed. Then we study consistency for the additive efficient normalization of semivalues. By the explicit interrelationship between the additive efficient normalization of a semivalue and the Shapley value in terms of the matrix approach, we define the $\mathcal{B}$-reduced game which is an extension of Sobolev's reduced game. It is shown that the $\mathcal{B}$-scaled game of the $\mathcal{B}$-reduced game is Sobolev's reduced game of the $\mathcal{B}$-scaled game. The additive efficient normalization of a semivalue is axiomatized as the unique value satisfying covariance, symmetry, and $\mathcal{B}$-consistency (reduced game property with respect to the $\mathcal{B}$-reduced game). Moreover, each linear, symmetric and efficient value is characterized by $\mathcal{B}$-consistency (respectively, linear consistency) and the λ-standardness for two-person games. The weighted extension forms of both the $\mathcal{B}$-reduced game and the linearly reduced game are also presented. For the class of linear, symmetric and efficient values, the weighted versions are shown to be equivalent to the standard versions of both the $\mathcal{B}$-reduced game and the linearly reduced game, respectively.

5.1   Reduced game and consistency

Let $\mathcal{G}$ denote the union of all game spaces $\mathcal{G}^N$, when the player set N is variable. In this chapter, a solution for a game refers to the universal game space $\mathcal{G}$. So, throughout this chapter the player set is always specified.

For the solution part, consistency is an important requirement of self-consistency or stability. It is an extremely general principle of fair division which states that if an allocation is fair, then every subgroup of claimants should agree to share the amount allotted to them fairly. This idea has been applied to a wide variety of allocation problems, including the apportionment of representation (Balinski and Young [7]), bankruptcy rules (Aumann and Maschler [6]), surplus sharing rules (Moulin [63]), bargaining problems (Harsanyi [40], Lensberg [56–58]), taxation (Young [104]), and economic exchange (Thomson [85]). For reviews of this literature see Thomson [86] and Young [105].

Generally speaking, the formulation of the consistency property for a solution is in terms of the solution itself and the so-called reduced games. A so-called reduced game is deducible from a given cooperative game by removing one or more players on the understanding that the removed players will be paid according to a specific principle (*e.g.*, a proposed payoff vector). The remaining players form the player set of the reduced game; the characteristic function of which is composed of the original characteristic function, the proposed payoff vector, and/or the solution in question. The consistency property for the solution states that if all the players are supposed to be paid according to a payoff vector in the solution set of the original game, then the players of the reduced game can achieve the corresponding payoff vector in the solution of the reduced game. In other words, there is no inconsistency in what the players of the reduced game can achieve, in either the original game or the reduced game.

Given any game $\langle N, v \rangle$, any coalition $T \subsetneq N$, and any payoff vector $x \in \mathbb{R}^N$, there are various ways to define a *reduced game* $\langle N \setminus T, v^x \rangle$ with respect to x, which is given in terms of the original characteristic function v and the payoff vector x. Note that the player set $N \setminus T$ of the reduced game is obtained here by removing the members of T from the original player set N. The reduced game $\langle N \setminus T, v^x \rangle$ should describe the following situation. Suppose

that all the players in N agree that the members of T will be paid according to the payoff vector x. Moreover, suppose that the non-members of T continue to cooperate with the members of T. Then the worth $v^x(S)$ of coalition $S \subseteq N \backslash T$ in the reduced game represents the total savings that the members of S may achieve subject to the foregoing two suppositions.

A solution Φ is said to be consistent (*i.e.*, possess the *reduced game property*) with respect to a specified type of reduced game, whenever the next condition is satisfied.

- *Consistency* (*reduced game property*): if $\langle N, v \rangle$ is a game, $T \subsetneqq N, T \neq \emptyset$ and $x \in \Phi(N, v)$, then $x^{N \backslash T} \in \Phi(N \backslash T, v^x)$.

The reduced game property says that if a payoff vector x is a point in the solution set $\Phi(N, v)$ of the original game $\langle N, v \rangle$, then the restriction $x^{N \backslash T}$ of x to any coalition $N \backslash T$ belongs to the solution set $\Phi(N \backslash T, v^x)$ of the corresponding reduced game $\langle N \backslash T, v^x \rangle$. In case the solution is a value, the reduced game condition requires that if the players are supposed to be paid according to the value, then there is no inconsistency in what the players of the reduced game will get, in either the original game or the reduced game. Thus, the reduced game property can be seen as a property of consistency.

Different solutions are consistent with respect to different reduced games (see Aumann and Drèze [3], Maschler and Owen [61], Maschler [62], Peleg [69], Driessen [27], Chang and Hu [12]). A detailed survey of almost all of them can be found in Thomson [87]. Here we review the most popular definitions of reduced games throughout the literature on cooperative game theory. Let $\langle N, v \rangle \in \mathcal{G}^N$ and $x \in \mathbb{R}^N$ be a payoff vector.

The first type was due to Davis and Maschler [14] for T being a single player coalition. Sobolev [78] used the definition repeatedly, to define the *max reduced game* $\langle N \backslash T, v^x \rangle$ as follows:

$$v^x_{N \backslash T}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \backslash T; \\ \max_{R \subseteq T} \{v(S \cup R) - x(R)\}, & \text{otherwise.} \end{cases} \tag{5.1.1}$$

Maschler and Davis observed that the pseudokernel satisfies the reduced game property with respect to the max reduced game. The development of this property of the kernel can be found in Maschler and Peleg [59], Maschler, Peleg,

and Shapley [60]. Later, Sobolev gave an axiomatization of the prenucleolus
by means of the max consistency.

The *complement reduced game* $\langle N \setminus T, v^x \rangle$ proposed by Moulin [63] is de-
fined as

$$v^x_{N \setminus T}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ v(S \cup T) - x(T), & \text{otherwise.} \end{cases} \qquad (5.1.2)$$

Moulin claimed that the equal allocation of nonseparable cost value of (1.2.4)
satisfies the complement consistency.

The *self-reduced game* $\langle N \setminus T, v^\Phi \rangle$ of $\langle N, v \rangle$ with respect to a value Φ is
proposed by Hart and Mas-Colell [42], and it is defined by

$$v^\Phi_{N \setminus T}(S) = v(S \cup T) - \sum_{i \in T} \Phi_i(S \cup T, v), \quad \text{for all } S \subseteq N \setminus T, S \neq \emptyset, \quad (5.1.3)$$

where $\Phi(S \cup T, v)$ denotes the value of the subgame $\langle S \cup T, v \rangle$. By means
of the self-consistency and one property with reference to two-person games
(see Lemma 5.3.7), Hart and Mas-Colell presented another characterization of
the Shapley value. Driessen and Radzik [28] extended Hart and Mas-Colell's
self-consistency to the class of linear, efficient and symmetric values. Another
well-known type of reduced game is the linearly reduced game, which is also
introduced with respect to the class of linear, symmetric and efficient values.
It will be studied in Section 5.4.

5.2 Matrix approach to the additive efficient normalization of semivalues

To define a semivalue (1.3.11) on the universal game space $\mathcal{G}$, we need a se-
quence of probability distributions $\{p^n\}_{n=1}^\infty$ all satisfying the normalization
condition (1.3.12). By adding a dummy player, any game $\langle N, v \rangle$ can be ex-
tended to the game $\langle N \cup \{d\}, v \rangle$ such that, according to the semivalue SE, each
player $i \in N$ in the new game should get the same outcome as in the original
game. So, in the framework of semivalues, there are some recursive relation-
ships between the probability distributions $p^n = (p^n_s)_{s=1}^n$ and $p^{n+1} = (p^{n+1}_s)_{s=1}^{n+1}$
given by

$$p^n_s = p^{n+1}_s + p^{n+1}_{s+1}, \quad s = 1, 2, \cdots, n. \qquad (5.2.1)$$

We call it the *inverse Pascal triangle condition* . It is easy to check that these relationships hold for the probability distribution collections of the Shapley value. As a result, for all $1 \le t \le n$, the probability distributions p^t are uniquely determined by p^n and the inverse Pascal triangle condition (5.2.1). Moreover, if the normalization condition (1.3.12) for $\mathcal{G}^N$ holds, then we derive from the inverse Pascal triangle relations that the normalization condition is satisfied for all $\mathcal{G}^T$, $T \subseteq N$.

In general, semivalues do not satisfy efficiency. In fact, the Shapley value is the unique efficient semivalue and this is a crucial requirement if one is looking for a solution that can be accepted by all the players. This motivated Ruiz *et al.* [73] to consider, for any semivalue SE, the *additive efficient normalization* ESE-value given by

$$ESE_i(N,v) = SE_i(N,v) + e, \quad \text{for all games } \langle N,v \rangle, \text{ and all } i \in N,$$

where the additive efficiency term $e = \frac{1}{n}\big[v(N) - \sum_{j \in N} SE_j(N,v)\big]$ is such that the value $ESE(N,v)$ is efficient. The $EANSC$-value (1.2.4) on $\mathcal{G}^N$ is the additive efficient normalization of the utopia vector (1.2.3), that is the semivalue with $p_n^n = 1$ and $p_s^n = 0$, for all $1 \le s \le n-1$.

The purpose of this section is to derive an explicit interrelationship between the additive efficient normalization of semivalues and the Shapley value, and secondly to establish a new axiomatization of the additive efficient normalization of semivalues. Because of the linearity of a semivalue and its additive efficient normalization, it is desirable to use the matrix approach to analyze the efficient normalization of any semivalue. Recall the notion of a coalitional matrix in order to represent any semivalue by means of a column-coalitional matrix in terms of the corresponding weight vector p^n as follows.

Definition 5.2.1. *Given a probability distribution $p^n = (p_s^n)_{s=1}^n$ and any game $\langle N,v \rangle$, the semivalue $SE(N,v)$ is represented by its representation matrix M^{SE} as:*

$$SE(N,v) = M^{SE} \cdot v,$$

where $M^{SE} = \big[M^{SE}\big]_{i \in N, S \in \Omega}$ is column-coalitional defined by

$$\big[M^{SE}\big]_{i,S} = \begin{cases} p_s^n, & \text{if } i \in S; \\ -p_{s+1}^n, & \text{if } i \notin S. \end{cases}$$

So, the semivalue $SE_i(N, v)$ of player i in the game $\langle N, v \rangle$ is just the inner product of the i-th row M_i^{SE} and the column vector v of coalitional worths. Particularly, for the Shapley value, we have the representation as given in Definition 2.3.2.

For a semivalue $SE(N, v)$ of any game $\langle N, v \rangle$, let $\vec{e} \in \mathbb{R}^N$ be the column vector with every entry equal to the additive efficiency term e. We call it the *additive efficiency vector* of $SE(N, v)$ and its matrix representation is given as follows.

Lemma 5.2.2. *Given any game* $\langle N, v \rangle$, *the additive efficiency vector* $\vec{e}$ *of a semivalue* $SE(N, v)$ *verifies the matrix representation* $\vec{e} = E \cdot v$, *where the matrix* $E = \big[E\big]_{i \in N, S \in \Omega}$ *is column-coalitional given by*

$$\big[E\big]_{i,S} = \begin{cases} \frac{n-s}{n} p_{s+1}^n - \frac{s}{n} p_s^n, & \text{if } S \neq N; \\ \frac{1}{n} - p_n^n, & \text{if } S = N. \end{cases}$$

Proof. By Definition 5.2.1,

$$\sum_{i \in N} SE_i(N, v) = \sum_{i \in N} \Big(\sum_{S \in \Omega} \big[M^{SE}\big]_{i,S} v(S) \Big) = \sum_{S \in \Omega} \Big(\sum_{i \in N} \big[M^{SE}\big]_{i,S} \Big) v(S)$$

$$= \sum_{\substack{S \subsetneq N \\ S \neq \emptyset}} \Big(\sum_{i \in S} \big[M^{SE}\big]_{i,S} + \sum_{i \notin S} \big[M^{SE}\big]_{i,S} \Big) v(S) + \sum_{i \in N} \big[M^{SE}\big]_{i,N} v(N)$$

$$= \sum_{\substack{S \subsetneq N \\ S \neq \emptyset}} \big[s p_s^n - (n - s) p_{s+1}^n\big] v(S) + n p_n^n v(N).$$

So, the additive efficiency term e of the semivalue $SE(N, v)$ is determined by

$$e = \frac{1}{n} \big[v(N) - \sum_{i \in N} SE_i(N, v)\big]$$

$$= \frac{1}{n} \sum_{\substack{S \subsetneq N \\ S \neq \emptyset}} \big[(n - s) p_{s+1}^n - s p_s^n\big] v(S) + \big(\frac{1}{n} - p_n^n\big) v(N).$$

Hence, $\vec{e} = E \cdot v$, where the column-coalitional matrix E is as given. $\qquad\square$

Now, for a given semivalue $SE(N, v)$, the algebraic representation of its additive efficient normalization $ESE(N, v)$ is as follows.

Theorem 5.2.3. *For any semivalue SE on $\mathcal{G}^N$, its additive efficient normalization ESE verifies the matrix representation $ESE(N, v) = M^{ESE} \cdot v$ for all games $\langle N, v \rangle$, where the matrix $M^{ESE} = \left[M^{ESE} \right]_{i \in N, S \in \Omega}$ is column-coalitional given by*

$$\left[M^{ESE} \right]_{i,S} = \begin{cases} \frac{n-s}{n} p_s^{n-1}, & \text{if } i \in S, \ S \neq N; \\ -\frac{s}{n} p_s^{n-1}, & \text{if } i \notin S; \\ \frac{1}{n}, & \text{if } S = N. \end{cases}$$

i.e., for all $i \in N$,

$$ESE_i(N, v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N \\ S \ni i}} \frac{n-s}{n} p_s^{n-1} \cdot v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} \frac{s}{n} p_s^{n-1} \cdot v(S). \quad (5.2.2)$$

Proof. For any game $\langle N, v \rangle$, by Definition 5.2.1 and Lemma 5.2.2, we have

$$ESE(N, v) = SE(N, v) + \vec{e} = M^{SE} \cdot v + E \cdot v = (M^{SE} + E) \cdot v.$$

It remains to prove the matrix equality $M^{SE} + E = M^{ESE}$. Let $i \in N$ and $S \in \Omega$. We distinguish three cases. If $S = N$, then

$$\left[M^{SE} \right]_{i,N} + \left[E \right]_{i,N} = p_n^n + \frac{1}{n} - p_n^n = \frac{1}{n} = \left[M^{ESE} \right]_{i,N}.$$

By the inverse Pascal triangle condition (5.2.1), if $i \in S$ and $S \neq N$, then

$$\left[M^{SE} \right]_{i,S} + \left[E \right]_{i,S} = p_s^n + \frac{n-s}{n} p_{s+1}^n - \frac{s}{n} p_s^n = \frac{n-s}{n} (p_s^n + p_{s+1}^n)$$
$$= \frac{n-s}{n} p_s^{n-1} = \left[M^{ESE} \right]_{i,S};$$

and if $i \notin S$, then

$$\left[M^{SE} \right]_{i,S} + \left[E \right]_{i,S} = -p_{s+1}^n + \frac{n-s}{n} p_{s+1}^n - \frac{s}{n} p_s^n = -\frac{s}{n} (p_s^n + p_{s+1}^n)$$
$$= -\frac{s}{n} p_s^{n-1} = \left[M^{ESE} \right]_{i,S}.$$

Therefore, the matrix equality $M^{SE} + E = M^{ESE}$ holds and so, (5.2.2) is valid for all $i \in N$. $\qquad \square$

By (5.2.2) and Corollary 2.2.8, we know that the additive efficient normalization of semivalues are included in the family of least square values. Now we treat the matrix representation of the interrelationship between the additive efficient normalization of a semivalue and the Shapley value by specifying the collection of nonnegative constants $\mathcal{B}$.

Theorem 5.2.4. $M^{ESE} = M^{Sh}B$, where $B = diag(b^n_{|S|})_{S \in \Omega}$ is the $\mathcal{B}$-scaling diagonal matrix such that $b^n_n = 1$ and $b^n_s = s\binom{n-1}{s}p^{n-1}_s$ for all $1 \le s \le n-1$.

Proof. We check the entry equalities $\left[M^{Sh}B\right]_{i,S} = \left[M^{ESE}\right]_{i,S}$ for all $i \in N$ and $S \in \Omega$. Since B is a diagonal matrix, we have

$$\left[M^{Sh}B\right]_{i,S} = \sum_{T \in \Omega} \left[M^{Sh}\right]_{i,T}[B]_{T,S} = \left[M^{Sh}\right]_{i,S}[B]_{S,S} = \left[M^{Sh}\right]_{i,S}b^n_s.$$

We distinguish three cases, too. Recall the formula (2.3.1) of the Shapley standard matrix M^{Sh}. If $S = N$, then

$$\left[M^{Sh}B\right]_{i,N} = \left[M^{Sh}\right]_{i,N} \cdot b^n_n = \frac{1}{n} \cdot 1 = \frac{1}{n} = \left[M^{ESE}\right]_{i,N};$$

if $i \in S$ and $S \ne N$, then

$$\left[M^{Sh}B\right]_{i,S} = \frac{(s-1)!(n-s)!}{n!} \cdot \frac{(n-1)!}{(s-1)!(n-s-1)!}p^{n-1}_s$$
$$= \frac{n-s}{n}p^{n-1}_s = \left[M^{ESE}\right]_{i,S};$$

and if $i \notin S$, then

$$\left[M^{Sh}B\right]_{i,S} = -\frac{s!(n-s-1)!}{n!} \cdot \frac{(n-1)!}{(s-1)!(n-s-1)!}p^{n-1}_s$$
$$= -\frac{s}{n}p^{n-1}_s = \left[M^{ESE}\right]_{i,S}.$$

We conclude that $M^{Sh}B = M^{ESE}$. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 5.2.5. Let $\mathcal{B} = \{b^n_s \mid n \ge 2, s = 1, 2, \cdots, n\}$ be the collection such that

$$b^n_n = 1 \quad and \quad b^n_s = s\binom{n-1}{s}p^{n-1}_s, \quad for\ all\ 1 \le s \le n-1. \qquad (5.2.3)$$

It is left to the reader to verify that the normalization condition (1.3.12) and the inverse Pascal triangle condition (5.2.1) for the probability distribution p^{n-1} can be reformulated as

$$\sum_{s=1}^{n} b^n_s = n, \quad and \quad b^n_s = \left(1 - \frac{s}{n}\right)b^{n+1}_s + \frac{s}{n}b^{n+1}_{s+1}, \quad s = 1, 2, \cdots, n-1, \ (5.2.4)$$

respectively. The sum of two subsequent probabilities in the inverse Pascal triangle condition (5.2.1) is replaced by a certain convex combination of the corresponding constants in $\mathcal{B}$. If $b_s^n = 1$ for all $1 \leq s \leq n$, we deduce from (5.2.4) that $b_s^k = 1$, for all $1 \leq k \leq n$ and all $1 \leq s \leq k$. It is the case that $ESE(N, v) = SE(N, v) = Sh(N, v)$.

Clearly, the additive efficient normalization of a semivalue satisfies efficiency, symmetry, and linearity. The result in Theorem 5.2.4 agrees with the results we have achieved in Theorem 4.1.5 and Corollary 4.1.6.

Corollary 5.2.6. *For any game $\langle N, v \rangle$, the additive efficient normalization $ESE(N, v)$ of the semivalue $SE(N, v)$ associated with p^n is the Shapley value of the corresponding $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$, where b^n is given by (5.2.3).*

By Corollary 2.3.4, the Shapley value $Sh(N, v)$ of a game $\langle N, v \rangle$ is of the following form

$$Sh_i(N, v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \big[v(S) - v(N \setminus S) \big], \quad \text{for all } i \in N.$$

We recite this formula in the complementary representation due to Driessen [24] like

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \big[v(N \setminus S) - v(S) \big], \quad \text{for all } i \in N.$$

Corollary 5.2.7. *For any game $\langle N, v \rangle$, the additive efficient normalization $ESE(N, v)$ of the semivalue $SE(N, v)$ associated with p^n is of the form*

$$ESE_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \big[b_{n-s}^n v(N \setminus S) - b_s^n v(S) \big], \quad \text{for all } i \in N,$$

where $b_n^n = 1$ and $b_s^n = s\binom{n-1}{s} p_s^{n-1}$, for all $1 \leq s \leq n-1$ as given in (5.2.3).

This formula is important for proving the main result of Theorem 5.3.8.

5.3 Consistency for the additive efficient normalization of semivalues

Formally speaking, a value Φ is said to be *consistent* or possesses the *reduced game property* with respect to a specified type of reduced game whenever the

following condition is satisfied: for all games $\langle N, v \rangle$, and all $S \in \Omega$,

$$\Phi_j(S, v_S^x) = \Phi_j(N, v), \quad \text{for all } j \in S, \text{ where } x = \Phi(N, v).$$

Firstly, we recall a type of reduced game considered by Sobolev [77] in order to axiomatize the Shapley value on the universal game space $\mathcal{G}$.

Definition 5.3.1 (Sobolev [77]). *Given a game $\langle N, v \rangle$, a player $i \in N$, and any payoff vector $x \in \mathbb{R}^N$, the corresponding reduced game $\langle N \setminus \{i\}, v_{N \setminus \{i\}}^x \rangle$ with respect to x is defined, for all $S \subseteq N \setminus \{i\}$, $S \neq \emptyset$, as*

$$v_{N \setminus \{i\}}^x(S) = \left(1 - \frac{s}{n-1}\right) v(S) + \frac{s}{n-1}\left[v(S \cup \{i\}) - x_i\right]. \tag{5.3.1}$$

Note that the worth (5.3.1) of any coalition is obtained as a convex combination of the worth of the coalition in the original game and the original worth of the coalition together with the single player minus the payoff x_i to the single player i for his participation. Sobolev showed the consistency for the Shapley value with respect to this reduced game $\langle N \setminus \{i\}, v_{N \setminus \{i\}}^x \rangle$ as

$$Sh_j(N \setminus \{i\}, v_{N \setminus \{i\}}^x) = Sh_j(N, v), \quad \text{for all } j \in N \setminus \{i\}, \text{ where } x = Sh(N, v).$$

Furthermore, the Shapley value on $\mathcal{G}$ is axiomatized by this property as follows.

Theorem 5.3.2 (Sobolev [77]). *The Shapley value is the unique value on $\mathcal{G}$ which possesses covariance, symmetry, and the reduced game property with respect to the reduced game of* (5.3.1).

In order to investigate consistency for the additive efficient normalization of semivalues, we introduce the following reduced game.

Definition 5.3.3. *Given a game $\langle N, v \rangle$, a player $i \in N$, a collection $\mathcal{B}$, and any payoff vector $x \in \mathbb{R}^N$, then the $\mathcal{B}$-reduced game $\langle N \setminus \{i\}, v_{N \setminus \{i\}}^{\mathcal{B},x} \rangle$ with respect to $\mathcal{B}$ and x is defined, for all $S \subseteq N \setminus \{i\}, S \neq \emptyset$, as*

$$v_{N \setminus \{i\}}^{\mathcal{B},x}(S) = \left(1 - \frac{s}{n-1}\right)\frac{b_s^n}{b_s^{n-1}} v(S) + \left(\frac{s}{n-1}\right)\frac{b_{s+1}^n}{b_s^{n-1}}\left[v(S \cup \{i\}) - \frac{x_i}{b_{s+1}^n}\right]. \tag{5.3.2}$$

By Remark 5.2.5, let the collection $\mathcal{B}$ verify the inverse Pascal triangle condition (5.2.4), *i.e.*,

$$b_s^{n-1} = \left(1 - \frac{s}{n-1}\right)b_s^n + \frac{s}{n-1}b_{s+1}^n, \quad s = 1, 2, \cdots, n-2.$$

Similar to Sobolev's reduced game (5.3.1), the $\mathcal{B}$-reduced game (5.3.2) is a certain convex combination of the worth of the coalition in the original game and the original worth of the coalition together with the single player minus the modified payoff $\frac{x_i}{b_{s+1}^n}$ to the single player i for his participation. Obviously, Sobolev's reduced game (5.3.1) is a particular $\mathcal{B}$-reduced game when $b_s^k = 1$ for all $1 \leq s \leq k$, and all $k \geq 2$. Furthermore, we have the following relationship between these two types of reduced games.

Proposition 5.3.4. *For any game $\langle N, v \rangle$, a player $i \in N$ and a collection $\mathcal{B}$, and any payoff vector $x \in \mathbb{R}^N$, the $\mathcal{B}$-scaled game (4.1.1) of the $\mathcal{B}$-reduced game (5.3.2) is Sobolev's reduced game (5.3.1) of the $\mathcal{B}$-scaled game (4.1.1), i.e.,*

$$\langle N \setminus \{i\}, \mathcal{B}(v_{N \setminus \{i\}}^{\mathcal{B},x}) \rangle = \langle N \setminus \{i\}, (\mathcal{B}v)_{N \setminus \{i\}}^x \rangle.$$

Proof. For any coalition $S \subseteq N \setminus \{i\}$, we have

$$
\begin{aligned}
(\mathcal{B}v)_{N \setminus \{i\}}^x(S) &= \Big(1 - \frac{s}{n-1}\Big)(\mathcal{B}v)(S) + \frac{s}{n-1}\big[(\mathcal{B}v)(S \cup \{i\}) - x_i\big] \\
&= \Big(1 - \frac{s}{n-1}\Big)b_s^n v(S) + \frac{s}{n-1}\big[b_{s+1}^n v(S \cup \{i\}) - x_i\big] \\
&= b_s^{n-1} v_{N \setminus \{i\}}^{\mathcal{B},x}(S) = \big(\mathcal{B}(v_{N \setminus \{i\}}^{\mathcal{B},x})\big)(S). \qquad \square
\end{aligned}
$$

Proposition 5.3.5. *The $\mathcal{B}$-reduced game is path-independent, i.e., for any game $\langle N, v \rangle$ with $n \geq 3$, any pair of players $i, j \in N, i \neq j$, and any payoff vector $x \in \mathbb{R}^N$, it holds for all $S \subseteq N \setminus \{i, j\}$, $S \neq \emptyset$, that*

$$\big(v_{N \setminus \{i\}}^{\mathcal{B},x}\big)_{N \setminus \{i,j\}}^{\mathcal{B},x}(S) = \big(v_{(N \setminus \{j\},x)}^{\mathcal{B},x}\big)_{N \setminus \{i,j\}}^{\mathcal{B},x}(S).$$

Proof. Fix $n \geq 3$. Starting with any n-person game $\langle N, v \rangle$, at first player $i \in N$ is removed and next the player $j \in N \setminus \{i\}$ is removed, taking into account their payoffs x_i, x_j respectively. By applying (5.3.2) twice,

$$\big(v_{N \setminus \{i\}}^{\mathcal{B},x}\big)_{N \setminus \{i,j\}}^{\mathcal{B},x}(N \setminus \{i,j\}) = \big(v_{N \setminus \{i\}}^{\mathcal{B},x}\big)(N \setminus \{i\}) - x_j = v(N) - x_i - x_j,$$

and for all coalitions $S \subsetneq N \setminus \{i, j\}$, it holds that

$$
\left(v_{N\setminus\{i\}}^{\mathcal{B},x}\right)_{N\setminus\{i,j\}}^{\mathcal{B},x}(S)
$$

$$
= \left(1 - \frac{s}{n-2}\right)\frac{b_s^{n-1}}{b_s^{n-2}}v_{N\setminus\{i\}}^{\mathcal{B},x}(S) + \frac{s}{n-2}\left[\frac{b_{s+1}^{n-1}}{b_s^{n-2}}v_{N\setminus\{i\}}^{\mathcal{B},x}(S \cup \{j\}) - \frac{x_j}{b_s^{n-2}}\right]
$$

$$
= \left(1 - \frac{s}{n-2}\right)\frac{b_s^{n-1}}{b_s^{n-2}}\left\{\left(1 - \frac{s}{n-1}\right)\frac{b_s^n}{b_s^{n-1}}v(S) + \frac{s}{n-1}\left[\frac{b_{s+1}^n}{b_s^{n-1}}v(S \cup \{i\}) - \frac{x_i}{b_s^{n-1}}\right]\right\}
$$

$$
+ \frac{s}{(n-2)}\frac{b_{s+1}^{n-1}}{b_s^{n-2}}\left\{\left(1 - \frac{s+1}{n-1}\right)\frac{b_{s+1}^n}{b_{s+1}^{n-1}}v(S \cup \{j\}) + \frac{s+1}{n-1}\left[\frac{b_{s+2}^n}{b_{s+1}^{n-1}}v(S \cup \{i,j\}) - \frac{x_i}{b_{s+1}^{n-1}}\right]\right\}
$$

$$
- \frac{s}{(n-2)}\frac{x_j}{b_s^{n-2}}
$$

$$
= \left(1 - \frac{s}{n-2}\right)\left(1 - \frac{s}{n-1}\right)\frac{b_s^n}{b_s^{n-2}}v(S) + \frac{s}{(n-2)}\frac{s+1}{(n-1)}\frac{b_{s+2}^n}{b_s^{n-2}}v(S \cup \{i,j\})
$$

$$
+ \left(1 - \frac{s}{n-2}\right)\frac{s}{(n-1)}\frac{b_{s+1}^n}{b_s^{n-2}}v(S \cup \{i\}) + \frac{s}{n-2}\left(1 - \frac{s+1}{n-1}\right)\frac{b_{s+1}^n}{b_s^{n-2}}v(S \cup \{j\})
$$

$$
- \frac{s}{(n-2)}\frac{x_i}{b_s^{n-2}} - \frac{s}{(n-2)}\frac{x_j}{b_s^{n-2}}.
$$

The path-independence property is deduced from the symmetry of the above expressions containing either i or j.   □

With respect to the additive efficient normalization of a semivalue, the collection $\mathcal{B}$ in the $\mathcal{B}$-reduced game (5.3.2) always is needed to verify the normalization condition and the inverse Pascal triangle condition in Remark 5.2.5. We consider the *p-version* of the $\mathcal{B}$-reduced game in terms of the probability distribution p^n. Since the collection $\mathcal{B}$ is given by $b_n^n = 1$ and $b_s^n = s\binom{n-1}{s}p_s^{n-1}$, for all $1 \leq s \leq n-1$ in (5.2.3), we have

$$
v_{N\setminus\{i\}}^{\mathcal{B},x}(S) = \begin{cases} v(N) - x_i, & \text{if } S = N \setminus \{i\}; \\ \frac{p_s^{n-1}}{p_s^{n-2}}v(S) \\ \quad + \frac{p_{s+1}^{n-1}}{p_s^{n-2}}\left[v(S \cup \{i\}) - \frac{1}{p_{s+1}^{n-1}} \cdot \frac{x_i}{(s+1)\binom{n-1}{s+1}}\right], & \text{otherwise.} \end{cases} \tag{5.3.3}
$$

For the $\mathcal{B}$-reduced game (5.3.2) to be well defined, the probability distribution $p^n = (p_s^n)_{s=1}^n$ should be positive. Since $p_s^{n-1} + p_{s+1}^{n-1} = p_s^{n-2}$, here the worth of any coalition in the $\mathcal{B}$-reduced game is also obtained as a convex combination of the worth of the coalition in the original game and the original worth of the coalition together with the single player minus the revised-payoff $\frac{1}{p_{s+1}^{n-1}} \cdot \frac{x_i}{(s+1)\binom{n-1}{s+1}}$ to the single player i for his participation.

Lemma 5.3.6. *The additive efficient normalization of a semivalue on $\mathcal{G}$ associated with a sequence of positive probability distributions $\{p^n\}_{n=1}^{\infty}$ satisfies the reduced game property with respect to the corresponding $\mathcal{B}$-reduced game (5.3.2).*

Proof. For a given game $\langle N, v \rangle$ and any player $i \in N$, by the relationship mentioned in Theorem 4.1.5 between the additive efficient normalization ESE-value of a semivalue and the Shapley value, we have, for all $j \in N \setminus \{i\}$,

$$ESE_j(N, v) = Sh_j(N, \mathcal{B}v), \quad \text{and}$$

$$ESE_j(N \setminus \{i\}, v_{N \setminus \{i\}}^{\mathcal{B},x}) = Sh_j(N \setminus \{i\}, \mathcal{B}(v_{N \setminus \{i\}}^{\mathcal{B},x})).$$

Hence, ESE satisfies the reduced game property with respect to the $\mathcal{B}$-reduced game (5.3.2) if and only if for all $j \in N \setminus \{i\}$,

$$Sh_j(N, \mathcal{B}v) = Sh_j(N \setminus \{i\}, \mathcal{B}(v_{N \setminus \{i\}}^{\mathcal{B},x})), \quad \text{where } x = ESE(N, v).$$

By the consistency of the Shapley value with respect to Sobolev's reduced game (5.3.1), applied to the $\mathcal{B}$-scaled game, we have

$$Sh_j(N, \mathcal{B}v) = Sh_j(N \setminus \{i\}, (\mathcal{B}v)_{N \setminus \{i\}}^{y}), \quad \text{for all } j \in N \setminus \{i\}, \text{ and } y = Sh(N, \mathcal{B}v).$$

Note that $y = Sh(N, \mathcal{B}v) = ESE(N, v) = x$. Together with Proposition 5.3.4, we obtain that for all $j \in N \setminus \{i\}$,

$$Sh_j(N, \mathcal{B}v) = Sh_j(N \setminus \{i\}, (\mathcal{B}v)_{N \setminus \{i\}}^{y}) = Sh_j(N \setminus \{i\}, \mathcal{B}(v_{N \setminus \{i\}}^{\mathcal{B},x})). \quad \square$$

For a value Φ on $\mathcal{G}$, the property of standardness for two-person games is defined as follows.

- *Standard for two-person games*: for any two person game $\langle \{i, j\}, v \rangle$,

$$\Phi_k(\{i, j\}, v) = v(\{k\}) + \frac{1}{2}\big(v(\{i, j\}) - v(\{i\}) - v(\{j\})\big), \text{ for } k \in \{i, j\}. \quad (5.3.4)$$

Lemma 5.3.7 (Driessen [27]). *If a value Φ on $\mathcal{G}$ satisfies covariance, and symmetry, then Φ is standard for two-person games.*

Theorem 5.3.8. *The additive efficient normalization of a semivalue on $\mathcal{G}$ associated with a sequence of positive probability distributions $\{p^n\}_{n=1}^{\infty}$ is the unique value on $\mathcal{G}$ that satisfies covariance, symmetry, and the reduced game property with respect to the corresponding $\mathcal{B}$-reduced game of (5.3.2).*

Proof. A semivalue as well as its additive efficient normalization possess the inessential game property. The linearity and inessential game property imply covariance of the additive efficient normalization. By Theorem 5.2.3 an entry in the matrix M^{ESE} is only related to the size of the coalition S and the relationship that the player i is a member or non-member of the coalition S. So the additive efficient normalization of a semivalue is symmetric. Together with Lemma 5.3.6, the additive efficient normalization of a semivalue possesses the three listed properties.

Now we treat the unicity proof. Let Φ be a value on $\mathcal{G}$ that satisfies covariance, symmetry, and the reduced game property with respect to the $\mathcal{B}$-reduced game of (5.3.2). We prove, by induction on n, that $\Phi(N, v) = ESE(N, v)$ for any n-person game $\langle N, v \rangle$. The case $n = 2$ follows from Lemma 5.3.7. Thus, let $\langle N, v \rangle \in \mathcal{G}$ with $n \geq 3$ and suppose that

$$\Phi(R, w) = ESE(R, w), \qquad \text{for all } \langle R, w \rangle \in \mathcal{G}, \text{ with } 2 \leq r < n.$$

Define $\langle N, u \rangle \in \mathcal{G}$ by

$$u(S) = v(S) - \sum_{j \in S} \Phi_j(N, v), \qquad \text{for all } S \subseteq N, S \neq \emptyset.$$

Since both values Φ and ESE possess covariance, we have

$$\Phi(N, u) = \vec{0}, \text{ and } ESE(N, u) = ESE(N, v) - \Phi(N, v).$$

So, the equality $\Phi(N, v) = ESE(N, v)$ is equivalent to $ESE_j(N, u) = 0$ for all $j \in N$. Let $i \in N$. By induction hypothesis and the reduced game property, we have

$$ESE_j(N\backslash\{i\}, u_{N\backslash\{i\}}^{\mathcal{B},x}) = \Phi_j(N\backslash\{i\}, u_{N\backslash\{i\}}^{\mathcal{B},x}) = \Phi_j(N, u) = 0, \text{ for all } j \in N\backslash\{i\},$$

where $x = \Phi(N, u)$. We denote $\Gamma_{ij} = \{S \in \Omega \mid S \ni i, S \not\ni j\}$ and $S^c = N \setminus S$ for all $S \in \Omega$. Put $r_s^n = \frac{s!(n-s-1)!}{n!}$, so that $r_s^n = r_{n-s-1}^n$. In the following, we deduce that $ESE_j(N, u) = 0$ from $ESE_j(N \setminus \{i\}, u_{N\backslash\{i\}}^{\mathcal{B},x}) = 0$.

By Corollary 5.2.7, we have

$$
\begin{aligned}
0 &= ESE_j(N \setminus \{i\}, u_{N \setminus \{i\}}^{\mathcal{B},x}) \\
&= \sum_{S \subseteq N \setminus \{i,j\}} r_s^{n-1} \Big[b_{n-1-s}^{n-1} u_{N \setminus \{i\}}^{\mathcal{B},x}((N \setminus \{i\}) \setminus S) - b_s^{n-1} u_{N \setminus \{i\}}^{\mathcal{B},x}(S) \Big] \\
&= \sum_{S \subseteq N \setminus \{i,j\}} r_s^{n-1} \Big[\frac{s}{n-1} b_{n-1-s}^n u\big((S \cup \{i\})^c\big) + (1 - \frac{s}{n-1}) b_{n-s}^n u(S^c) \\
&\quad - (1 - \frac{s}{n-1}) b_s^n u(S) - \frac{s}{n-1} b_{s+1}^n u(S \cup \{i\}) \Big] \\
&= \frac{1}{n-1} \sum_{S \subseteq N \setminus \{i,j\}} r_s^{n-1} \Big\{ s \big[b_{n-1-s}^n u\big((S \cup \{i\})^c\big) - b_{s+1}^n u(S \cup \{i\}) \big] \\
&\quad - (n - s - 1) \big[b_s^n u(S) - b_{n-s}^n u(S^c) \big] \Big\} \\
&= \frac{1}{n-1} \sum_{T \in \Gamma_{ij}} (t-1) r_{t-1}^{n-1} \Big[b_{n-t}^n u\big(T^c\big) - b_t^n u(T) \Big] \\
&\quad + \frac{n}{n-1} \sum_{S \subseteq N \setminus \{i,j\}} r_s^n \Big[b_{n-s}^n u(S^c) - b_s^n u(S) \Big] \\
&= \frac{n}{n-1} \sum_{T \in \Gamma_{ij}} r_t^n \Big[b_{n-t}^n u(T^c) - b_t^n u(T) \Big] \\
&\quad + \frac{n}{n-1} \sum_{S \subseteq N \setminus \{i,j\}} r_s^n \Big[b_{n-s}^n u(S^c) - b_s^n u(S) \Big] \\
&\quad + \frac{1}{n-1} \sum_{T \in \Gamma_{ij}} \big((t-1) r_{t-1}^{n-1} - n r_t^n \big) \Big[b_{n-t}^n u(T^c) - b_t^n u(T) \Big] \\
&= \frac{n}{n-1} ESE_j(N, u) - \frac{1}{n-1} \sum_{T \in \Gamma_{ij}} r_{t-1}^{n-1} \Big[b_{n-t}^n u(T^c) - b_t^n u(T) \Big].
\end{aligned}
$$

Thus,

$$
ESE_j(N, u) = \frac{1}{n} \sum_{T \in \Gamma_{ij}} r_{t-1}^{n-1} \Big[b_{n-t}^n u(T^c) - b_t^n u(T) \Big], \quad \text{for all } i, j \in N,\ i \neq j.
$$

By interchanging the players i and j, we obtain that for all $i, j \in N, i \neq j$,

using $S = T^c$,

$$
\begin{aligned}
ESE_i(N, u) &= \frac{1}{n} \sum_{T \in \Gamma_{ji}} r_{t-1}^{n-1} \left[b_{n-t}^n u(T^c) - b_t^n u(T) \right] \\
&= \frac{1}{n} \sum_{S \in \Gamma_{ij}} r_{n-s-1}^{n-1} \left[b_s^n u(S) - b_{n-s}^n u(S^c) \right] \\
&= -\frac{1}{n} \sum_{S \in \Gamma_{ij}} r_{s-1}^{n-1} \left[b_{n-s}^n u(S^c) - b_s^n u(S) \right] \\
&= -ESE_j(N, u).
\end{aligned}
$$

From this and $n \geq 3$, we conclude that, for any three players $i, j, k \in N$, $ESE_i(N, u) = -ESE_k(N, u) = ESE_j(N, u)$. Therefore, $-ESE_j(N, u) = ESE_j(N, u)$ and, $ESE_j(N, u) = 0$, for all $j \in N$. $\square$

Remark 5.3.9. *For characterizing the additive efficient normalization of semi-values, the collection $\mathcal{B}$ of the $\mathcal{B}$-reduced game of (5.3.2) should be positive and verifies the normalization condition and the inverse Pascal triangle condition:*

$$
\sum_{s=1}^n b_s^n = n, \quad and \quad b_s^n = \left(1 - \frac{s}{n}\right) b_s^{n+1} + \frac{s}{n} b_{s+1}^{n+1}, \quad s = 1, 2, \cdots, n-1. \quad (5.3.5)
$$

5.4   Linear consistency and Sobolev's consistency

For any game $\langle N, v \rangle$, any player $i \in N$, and any payoff vector $x \in \mathbb{R}^N$, the *linearly reduced game* $\langle N \setminus \{i\}, v_{N \setminus \{i\}}^x \rangle$ with respect to x is defined as

$$
v_{N \setminus \{i\}}^x(S) = \begin{cases} v(N) - x_i, & \text{if } S = N \setminus \{i\}; \\ (1 - w_{n,s})v(S) + w_{n,s}\left[v(S \cup \{i\}) - x_i\right], & \text{otherwise,} \end{cases} \quad (5.4.1)
$$

where $w_{n,s} \in [0, 1]$, $1 \leq s \leq n-2$, are weight coefficients. The subgame and the complement reduced game are particular cases of (5.4.1) when $w_{n,s} = 0$, respectively $w_{n,s} = 1$ for all $1 \leq s \leq n-2$. Sobolev's reduced game (5.3.1) is also a particular linearly reduced game with $w_{n,s} = \frac{s}{n-1}$. The p-version $\mathcal{B}$-reduced game (5.3.3) of the additive efficient normalization of semivalues is a slight adaptation of the linearly reduced game with $w_{n,s} = \frac{p_{s+1}^{n-1}}{p_s^{n-2}}$ and the revised-payoff $\frac{1}{p_{s+1}^{n-1}} \cdot \frac{x_i}{(s+1)\binom{n-1}{s+1}}$. They coincide on Sobolev's reduced game.

Concerning any type of reduced game, the worth of its grand coalition $N\backslash T$ is equal to $v(N) - x(T)$. Such a definition preserves efficiency of a solution vector x in any reduced game. Therefore, the sequel of linearly reduced games also follows this procedure. Let $m^n = (m_s^n)_{s=1}^n$ be a weight collection. For any game $\langle N, v\rangle$, any $T \subsetneq N$, $T \ne \emptyset$, and any given payoff vector $x \in \mathbb{R}^N$, the *linearly reduced game* $\langle N \setminus T, v^x\rangle$ with respect to x (called *m-reduced game* by Ruiz, Valenciano and Zarzuelo [73]), is defined as

$$v_{N\backslash T}^x(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \sum\limits_{R \subseteq T} \dfrac{m_{s+r}^n}{\sigma_{t,s}^n}\big[v(S \cup R) - x(R)\big], & \text{otherwise,} \end{cases} \tag{5.4.2}$$

where $\sigma_{t,s}^n = \sum\limits_{r=0}^t \binom{t}{r} m_{s+r}^n$. A value is *linearly consistent (i.e., m-consistent)* if it satisfies the reduced game property with respect to the m-reduced game (5.4.2).

A weight collection $m = \{m^n\}_{n=2}^\infty$ is *consistent* when the weights, normalized according to $\sum\limits_{s=1}^{n-1} \binom{n-2}{s-1} m_s^n = 1$, verify the relation $m_s^n = m_s^{n+1} + m_{s+1}^{n+1}$, for all n and all $1 \le s \le n - 1$.

Proposition 5.4.1. *For each consistent weight collection m, the corresponding least square value LS^m on $\mathcal{G}$ is the additive efficient normalization of the semivalue with $p_s^{n-1} = m_s^n$, for all n and all $1 \le s \le n - 1$, and vice versa.*

Proof. If m is consistent, then by (1.3.13), $\sigma = \sum\limits_{s=1}^{n-1} \binom{n-2}{s-1} m_s^n = 1$, while for any game $\langle N, v\rangle$, we have

$$LS_i^m(N, v) = \frac{v(N)}{n} + \frac{1}{n}\Big[\sum_{\substack{S \subsetneq N \\ S \ni i}} (n-s)m_s^n \cdot v(S) - \sum_{\substack{S \not\ni i \\ S \ne \emptyset}} s m_s^n \cdot v(S) \Big], \quad \text{for all } i \in N.$$

Let $p_s^{n-1} = m_s^n$, for all $n \ge 2$ and all $1 \le s \le n - 1$. Then $\sum\limits_{s=1}^n \binom{n-1}{s-1} p_s^n = 1$ and $p_s^n = p_s^{n+1} + p_{s+1}^{n+1}$, for all $n \ge 1$ and all $1 \le s \le n$. So the sequence $p = \{p^n\}_{n=1}^\infty$ satisfies the normalization condition and the inverse Pascal condition. Meanwhile,

$$LS_i^m(N, v) = \frac{v(N)}{n} + \frac{1}{n}\Big[\sum_{\substack{S \subsetneq N \\ S \ni i}} (n-s)p_s^{n-1} \cdot v(S) - \sum_{\substack{S \not\ni i \\ S \ne \emptyset}} s p_s^{n-1} \cdot v(S) \Big], \quad \text{for all } i \in N.$$

By (5.2.2) in Theorem 5.2.3, we conclude that LS^m is the additive efficient normalization of the semivalue with $p_s^{n-1} = m_s^n$ for all $n \geq 2$ and all $1 \leq s \leq n-1$, and vice versa.   □

The consistency of the weight collection m is equivalent to the normalization condition and the inverse Pascal condition of the corresponding sequence p of probability distributions. Moreover, for the family of least square values, Ruiz *et al.* [73] showed the following.

Theorem 5.4.2 (Ruiz *et al.* [73]). *The least square value LS^m on $\mathcal{G}$ associated with the weight collection m is linearly consistent if and only if m is a consistent weight collection.*

Theorem 5.4.3 (Ruiz *et al.* [73]). *For each consistent weight collection m, the corresponding least square value LS^m on $\mathcal{G}$ is the unique value that satisfies linear consistency and the standardness for two-person games .*

By Proposition 5.4.1 and Theorem 5.4.2, we obtain directly the following relation between the family of least square values and the additive efficient normalization of semivalues.

Theorem 5.4.4 (Ruiz *et al.* [73]). *A least square value LS^m on $\mathcal{G}$ is linearly consistent if and only if it is the additive efficient normalization of a semivalue.*

In comparison with (5.4.2), we prefer the simplified representation for the linearly reduced game given by Yanovskaya and Driessen [102] as

$$
v_{N \setminus T}^x(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \sum_{R \subseteq T} \alpha_{t,s,r}^n \left[v(S \cup R) - x(R) \right], & \text{otherwise,} \end{cases} \tag{5.4.3}
$$

where $\alpha_{t,s,r}^n$ are arbitrary non-negative real numbers such that $\sum_{r=0}^{t} \binom{t}{r} \alpha_{t,s,r}^n = 1$, for all $1 \leq t \leq n-1$, and all $1 \leq s \leq n-t-1$.

Yanovskaya and Driessen [102] classified linearly consistent values to be linear, symmetric and efficient, provided that some kind of standardness for two-person games applies.

Theorem 5.4.5 (Yanovskaya and Driessen [102]). *Let Φ be a value on $\mathcal{G}$ which is linearly consistent and λ-standard for two-person games (for some*

$\lambda \in \mathbb{R}$). *Then the value* Φ *is linear, symmetric, as well as efficient. Moreover,* Φ *is the unique value on* $\mathcal{G}$ *satisfying linear consistency and the* λ-*standardness for two-person games.*

For a value Φ on $\mathcal{G}$, the property of λ-standardness for two-person games (where $\lambda \in \mathbb{R}$) is defined as follows.

- λ-*standard for two-person games*: for any two-person game $\langle \{i,j\}, v \rangle$,

$$\Phi_k(\{i,j\},v) = \lambda v(\{k\}) + \frac{1}{2}\big[v(\{i,j\}) - \lambda v(\{i\}) - \lambda v(\{j\})\big], \text{ for } k \in \{i,j\}. \quad (5.4.4)$$

Obviously, the standardness for two-person games is a particular λ-standardness with $\lambda = 1$. In their axiomatic approach, Yanovskaya and Driessen [102] derived a formula for any linearly consistent value Φ satisfying λ-standardness for two-person games, as follows: for any game $\langle N, v \rangle$, and all $i \in N$,

$$\Phi_i(N,v) = \frac{v(N)}{n} + \frac{\lambda}{n}\Big[\sum_{\substack{S \subsetneq N \\ S \ni i}} (n-s)\alpha^n_{n-2,1,s-1}v(S) - \sum_{S \not\ni i} s\alpha^n_{n-2,1,s-1}v(S)\Big]. \quad (5.4.5)$$

For future purposes, we introduce the following generalization of Sobolev's reduced game (5.3.1).

Definition 5.4.6. *Given a game* $\langle N,v \rangle$, *a coalition* $T \subsetneq N$, $T \neq \emptyset$, *and any payoff vector* $x \in \mathbb{R}^N$, *then the reduced game* $\langle N \setminus T, v^x_{N \setminus T} \rangle$ *with respect to* x *is defined as*

$$v^x_{N \setminus T}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \displaystyle\sum_{R \subseteq T} \frac{s\binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}}\big[v(S \cup R) - x(R)\big], & \text{otherwise.} \end{cases} \quad (5.4.6)$$

This reduced game is also a special linearly reduced game, which involves a new type of probability distribution. Because of the following calculations

$$\sum_{R \subseteq T} \frac{s\binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}} = \sum_{r=0}^{t} \frac{s\binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}}\binom{t}{r} = \sum_{r=0}^{t} \frac{\binom{s+r-1}{r}\binom{n-s-r-1}{t-r}}{\binom{n-1}{t}} = 1. \quad (5.4.7)$$

The proof of the latter equality proceeds by induction on n for all cases of t, $1 \le t \le n-1$, together with the combinatorial property $\binom{n}{t} + \binom{n}{t-1} = \binom{n+1}{t}$. This combinatorial equality can be interpreted as the number of possibilities

to put t identical balls into $n - t$ different boxes, either directly or indirectly, to enumerate by addition, for every $0 \leq r \leq t$, the possibilities to put r identical balls into s different boxes and the remaining $t - r$ identical balls into the remaining $n - t - s$ different boxes.

The worth $v^x_{N \setminus T}(S)$ of coalition S in the reduced game (5.4.6) is interpreted as the expectation of coalition S in the following probabilistic setting. On the one hand, it is assumed that the remaining players in $(N \setminus T) \setminus S$ are not present, on the other that players in T act as possible partners for coalition S. If coalition S forms a partnership with set R from T, the coalition S is conceded to benefit by what is left of the worth $v(S \cup R)$ of the resulting partnership, given that their partners j, $j \in R$, have to be paid x_j according to the proposed payoff vector x. The probability that S forms a partnership with set R from T is equal to $\frac{s \binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}}$.

Theorem 5.4.7. *Given a game $\langle N, v \rangle$, a coalition $T \subsetneq N$, $T \neq \emptyset$, and any payoff vector $x \in \mathbb{R}^N$, the reduced game (5.4.6) with respect to x is the generalized Sobolev's reduced game (5.3.1), i.e.,*

$$\langle N \setminus T, v^x_{N \setminus T} \rangle = \langle N \setminus T, (v^x_{N \setminus (T \setminus \{i\})})^x_{N \setminus T} \rangle, \quad \text{for all } i \in T.$$

Consequently, the Shapley value satisfies the reduced game property with respect to the reduced game (5.4.6).

Proof. For showing the reduced game (5.4.6) is the generalized Sobolev's reduced game, we apply Definition 5.3.1 repeatedly to achieve (5.4.6). Obviously, if $T = \{i\}$ for some $i \in N$, then the reduced game (5.4.6) agrees with the Sobolev's reduced game. By induction hypothesis, suppose that the repetition holds up to the removal of coalition $T \setminus \{i\}$, for all $i \in T$, *i.e.,*

$$v^x_{N \setminus (T \setminus \{i\})}(S) = \begin{cases} v(N) - x(T \setminus \{i\}), & \text{if } S = N \setminus (T \setminus \{i\}); \\ \displaystyle\sum_{R \subseteq T \setminus \{i\}} \frac{s \binom{n-t}{s}}{(s+r)\binom{n-1}{s+r}} \big[v(S \cup R) - x(R)\big], & \text{otherwise.} \end{cases}$$

Applying (5.3.1) to the reduced game $\langle N \setminus (T \setminus \{i\}), v^x_{N \setminus (T \setminus \{i\})} \rangle$, we obtain, for all $S \subseteq N \setminus T, S \neq \emptyset$, that

$$(v^x_{N \setminus (T \setminus \{i\})})^x_{N \setminus T}(S) = \big(1 - \frac{s}{n-t}\big) v^x_{N \setminus (T \setminus \{i\})}(S) + \frac{s}{n-t} \big[v^x_{N \setminus (T \setminus \{i\})}(S \cup \{i\}) - x_i\big].$$

If $S = N \setminus T$, then

$$(v^x_{N \setminus (T \setminus \{i\})})^x_{N \setminus T}(N \setminus T) = v^x_{N \setminus (T \setminus \{i\})}(N \setminus (T \setminus \{i\}) \cup \{i\}) - x_i$$

$$= v(N) - x(T \setminus \{i\}) - x_i = v(N) - x(T).$$

Otherwise, if $S \subsetneqq N \setminus T$, $S \neq \emptyset$, then

$$(v^x_{N \setminus (T \setminus \{i\})})^x_{N \setminus T}(S) = \left(1 - \frac{s}{n-t}\right) \sum_{R \subseteq T \setminus \{i\}} \frac{s \binom{n-t}{s}}{(s+r) \binom{n-1}{s+r}} \left[v(S \cup R) - x(R)\right]$$

$$+ \frac{s}{n-t} \left\{ \sum_{R \subseteq T \setminus \{i\}} \frac{(s+1) \binom{n-t}{s+1}}{(s+1+r) \binom{n-1}{s+1+r}} \left[v(S \cup \{i\} \cup R) - x(R)\right] - x_i \right\}$$

$$= \sum_{R \subseteq T \setminus \{i\}} \frac{s \binom{n-t-1}{s}}{(s+r) \binom{n-1}{s+r}} \left[v(S \cup R) - x(R)\right]$$

$$+ \left\{ \sum_{R \subseteq T \setminus \{i\}} \frac{s \binom{n-t-1}{s}}{(s+1+r) \binom{n-1}{s+1+r}} \left[v(S \cup \{i\} \cup R) - x(R)\right] - \frac{s}{n-t} x_i \right\}$$

$$= \sum_{\substack{K \subseteq T \\ K \not\ni i}} \frac{s \binom{n-t-1}{s}}{(s+k) \binom{n-1}{s+k}} \left[v(S \cup K) - x(K)\right]$$

$$+ \sum_{\substack{K \subseteq T \\ K \ni i}} \frac{s \binom{n-t-1}{s}}{(s+k) \binom{n-1}{s+k}} \left[v(S \cup K) - x(K) + x_i\right] - \frac{s}{n-t} x_i$$

$$= \sum_{K \subseteq T} \frac{s \binom{n-t-1}{s}}{(s+k) \binom{n-1}{s+k}} \left[v(S \cup K) - x(K)\right].$$

The last equality is based on

$$\sum_{\substack{K \subseteq T \\ K \ni i}} \frac{s \binom{n-t-1}{s}}{(s+k) \binom{n-1}{s+k}} = \frac{s}{n-t}, \quad \text{i.e.,} \quad \sum_{k=1}^{t} \binom{s+k-1}{k-1} \binom{n-s-k-1}{t-k} = \binom{n-1}{t-1}.$$

The inductive proof of the latter equality proceeds similar to (5.4.7). Consequently, the Shapley value satisfies the reduced game property with respect to the generalized Sobolev's reduced game (5.4.6). $\qquad\square$

5.5 $\mathcal{B}$-consistency and path-independently linear consistency

For a linear, symmetric and efficient value Φ on $\mathcal{G}$, by Theorem 4.1.5 and Corollary 4.1.6, there exists a collection $\mathcal{B} = \{b_s^n \mid n \geq 2, s = 1, 2, \cdots, n\}$ with $b_n^n = 1$ for all $n \geq 2$, such that $\Phi(N, v) = Sh(N, \mathcal{B}v)$ for all games $\langle N, v \rangle$. By the path-independence of the $\mathcal{B}$-reduced game as stated in Proposition 5.3.5, we are going to define the $\mathcal{B}$-reduced game $\langle N \setminus T, v_{N\setminus T}^{\mathcal{B},x} \rangle$, by repeatedly applying (5.3.2), as follows.

Definition 5.5.1. *Given a game $\langle N, v \rangle$, a coalition $T \subsetneq N$, $T \neq \emptyset$, a collection $\mathcal{B}$, and any payoff vector $x \in \mathbb{R}^N$, then the $\mathcal{B}$-reduced game $\langle N \setminus T, v_{N\setminus T}^{\mathcal{B},x} \rangle$ with respect to $\mathcal{B}$ and x is defined as*

$$v_{N\setminus T}^{\mathcal{B},x}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \displaystyle\sum_{R \subseteq T} \frac{s\binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}} \frac{b_{s+r}^n}{b_s^{n-t}} \left[v(S \cup R) - \frac{x(R)}{b_{s+r}^n} \right], & \text{otherwise.} \end{cases} \tag{5.5.1}$$

The relationship presented in Proposition 5.3.4 can be extended to the generalized reduced games as follows.

Proposition 5.5.2. *For any game $\langle N, v \rangle$, a player $i \in N$ and a collection $\mathcal{B}$, the $\mathcal{B}$-scaled game of the $\mathcal{B}$-reduced game (5.5.1) is the generalized Sobolev's reduced game (5.4.6) of the $\mathcal{B}$-scaled game , i.e.,*

$$\langle N \setminus T, \mathcal{B}(v_{N\setminus T}^{\mathcal{B},x}) \rangle = \langle N \setminus T, (\mathcal{B}v)_{N\setminus T}^x \rangle, \quad \text{for all } x \in \mathbb{R}^N.$$

Proof. Recall that $b_m^m = 1$ for all $m \geq 2$. So,

$$\left(\mathcal{B}(v_{N\setminus T}^{\mathcal{B},x}) \right)(N \setminus T) = v_{N\setminus T}^{\mathcal{B},x}(N \setminus T) = v(N) - x(T)$$

$$= (\mathcal{B}v)(N) - x(T) = (\mathcal{B}v)_{N\setminus T}^x(N \setminus T).$$

For all $S \subsetneq N \setminus T$, $S \neq \emptyset$, we have

$$(\mathcal{B}v)_{N\setminus T}^x(S) = \sum_{R \subseteq T} \frac{s\binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}} \left[(\mathcal{B}v)(S \cup R) - x(R) \right]$$

$$= \sum_{R \subseteq T} \frac{s\binom{n-t-1}{s}}{(s+r)\binom{n-1}{s+r}} \left[b_{s+r}^n v(S \cup R) - x(R) \right]$$

$$= b_s^{n-t} v_{N\setminus T}^{\mathcal{B},x}(S) = \left(\mathcal{B}(v_{N\setminus T}^{\mathcal{B},x}) \right)(S). \qquad \square$$

Similar to Lemma 5.3.6, we can prove the following theorem.

Theorem 5.5.3. *For a given collection $\mathcal{B}$, the corresponding linear, symmetric and efficient value Φ on $\mathcal{G}$ satisfies $\mathcal{B}$-consistency, i.e., the reduced game property with respect to the corresponding $\mathcal{B}$-reduced game (5.5.1).*

Proof. For any game $\langle N, v \rangle$ and any $T \subsetneq N, T \neq \emptyset$, we have , by Theorem 4.1.5, for all $j \in N \setminus T$,

$$\Phi_j(N, v) = Sh_j(N, \mathcal{B}v) \quad \text{and} \quad \Phi_j(N \backslash T, v_{N \backslash T}^{\mathcal{B}, x}) = Sh_j(N \backslash T, \mathcal{B}(v_{N \backslash T}^{\mathcal{B}, x})).$$

Hence, Φ satisfies $\mathcal{B}$-consistency with respect to the $\mathcal{B}$-reduced game (5.5.1) if and only if for all $j \in N \setminus T$,

$$Sh_j(N, \mathcal{B}v) = Sh_j(N \setminus T, \mathcal{B}(v_{N \backslash T}^{\mathcal{B}, x})), \quad \text{where } x = \Phi(N, v).$$

By the consistency of the Shapley value with respect to the generalized Sobolev's reduced game (5.4.6), applied to the $\mathcal{B}$-scaled game, we have

$$Sh_j(N, \mathcal{B}v) = Sh_j(N \setminus T, (\mathcal{B}v)_{N \backslash T}^y) \quad \text{for all } j \in N \setminus T, \text{ and } y = Sh(N, \mathcal{B}v).$$

Note that $y = Sh(N, \mathcal{B}v) = \Phi(N, v) = x$. Together with Proposition 5.5.2, we obtain that for all $j \in N \setminus T$,

$$Sh_j(N, \mathcal{B}v) = Sh_j(N \backslash T, (\mathcal{B}v)_{N \backslash T}^y) = Sh_j(N \setminus T, \mathcal{B}(v_{N \backslash T}^{\mathcal{B}, x})). \qquad \square$$

For characterizing the class of linear, symmetric and efficient values, and revealing the relation between the linearly reduced game and the $\mathcal{B}$-reduced game, similar to (5.3.3), we present the *p-version $\mathcal{B}$*-reduced game as follows.

$$v_{N \backslash T}^{\mathcal{B}, x}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \sum_{R \subseteq T} \frac{p_{s+r}^{n-1}}{p_s^{n-t-1}} \left[v(S \cup R) - \frac{1}{p_{s+r}^{n-1}} \cdot \frac{x(R)}{(s+r)\binom{n-1}{s+r}} \right], & \text{otherwise,} \end{cases} \quad (5.5.2)$$

where $p_s^{n-1} = \frac{b_s^n}{s\binom{n-1}{s}}$, for all n and $1 \leq s \leq n-1$, is from (5.2.3). A value is also said to be $\mathcal{B}$-*consistent* if it satisfies the reduced game property with respect to the p-version $\mathcal{B}$-reduced game (5.5.2). By this p-version, together with the λ-standardness for two-person games, we can derive inductively the formula of a linear, symmetric and efficient value.

Theorem 5.5.4. *For a given collection $\mathcal{B}$, the corresponding linear, symmetric and efficient value Φ on $\mathcal{G}$ is the unique value that satisfies the λ-standardness for two-person games and $\mathcal{B}$-consistency.*

Proof. For a given collection $\mathcal{B}$, by Corollary 2.2.8, it is to easy to check that the corresponding linear, symmetric and efficient value Φ on $\mathcal{G}$ is λ-standardness for two-person games. By Lemma 5.5.3, we know that Φ is $\mathcal{B}$-consistent. Now we show that, for a value Φ, the $\mathcal{B}$-consistency, together with the λ-standardness for two-person games, imply linearity, symmetry and efficiency, as well as the uniqueness of the value. The procedure is similar to Yanovskaya and Driessen's proof of Theorem 5.4.5.

Clearly, the λ-standardness implies the efficiency of Φ for two-person games. Generally speaking, the $\mathcal{B}$-consistency for Φ implies the efficiency of Φ, the inductive proof of which proceeds as follows. For any game $\langle N, v \rangle$ with $n \geq 3$, and any $i \in N$,

$$
\begin{aligned}
\sum_{j \in N} \Phi_j(N, v) &= \Phi_i(N, v) + \sum_{j \in N \setminus \{i\}} \Phi_j(N, v) \\
&= \Phi_i(N, v) + \sum_{j \in N \setminus \{i\}} \Phi_j(N \setminus \{i\}, v^{\mathcal{B},x}_{N \setminus \{i\}}) \\
&= \Phi_i(N, v) + v^{\mathcal{B},x}_{N \setminus \{i\}}(N \setminus \{i\}) = v(N), \quad \text{where } x = \Phi(N, v).
\end{aligned}
$$

Fix a two-person coalition $\Gamma = \{i, j\}, i \neq j$, and consider the associated two-person $\mathcal{B}$-reduced game $\langle \Gamma, v^{\mathcal{B},x}_\Gamma \rangle$. Since Φ is efficient and $\mathcal{B}$-consistent, together with the property of λ-standardness for two-person games, we obtain that

$$
\Phi_i(N, v) + \Phi_j(N, v) = \Phi_i(\Gamma, v^{\mathcal{B},x}_\Gamma) + \Phi_j(\Gamma, v^{\mathcal{B},x}_\Gamma) = v^{\mathcal{B},x}_\Gamma(\{i, j\}),
$$

$$
\Phi_i(N, v) = \Phi_i(\Gamma, v^{\mathcal{B},x}_\Gamma) = \lambda v^{\mathcal{B},x}_\Gamma(\{i\}) + \tfrac{1}{2} \big[v^{\mathcal{B},x}_\Gamma(\{i, j\}) - \lambda v^{\mathcal{B},x}_\Gamma(\{i\}) - \lambda v^{\mathcal{B},x}_\Gamma(\{j\}) \big].
$$

From this, we derive the following equality

$$
\Phi_i(N, v) = \Phi_j(N, v) + \lambda \big[v^{\mathcal{B},x}_\Gamma(\{i\}) - v^{\mathcal{B},x}_\Gamma(\{j\}) \big]. \tag{5.5.3}
$$

Furthermore, by (5.5.2) of the two-person $\mathcal{B}$-reduced game, we have

$$
\begin{aligned}
v_\Gamma^{\mathcal{B},x}(\{i\}) - v_\Gamma^{\mathcal{B},x}(\{j\}) &= \sum_{R \subseteq N \setminus \{i,j\}} \frac{p_{1+r}^{n-1}}{p_1^1} \left[v(R \cup \{i\}) - \frac{1}{p_{1+r}^{n-1}} \cdot \frac{x(R)}{(1+r)\binom{n-1}{1+r}} \right] \\
&\quad - \sum_{R \subseteq N \setminus \{i,j\}} \frac{p_{1+r}^{n-1}}{p_1^1} \left[v(R \cup \{j\}) - \frac{1}{p_{1+r}^{n-1}} \cdot \frac{x(R)}{(1+r)\binom{n-1}{1+r}} \right] \\
&= \sum_{\substack{T \subseteq N \setminus \{j\} \\ T \ni i}} \frac{p_t^{n-1}}{p_1^1} v(T) - \sum_{\substack{T \subseteq N \setminus \{i\} \\ T \ni j}} \frac{p_t^{n-1}}{p_1^1} v(T).
\end{aligned}
$$

Hence, for every pair $\{i,j\}$, $i \neq j$, of players, (5.5.3) reduces to

$$
\Phi_i(N,v) = \Phi_j(N,v) + \frac{\lambda}{p_1^1} \left[\sum_{\substack{T \subseteq N \setminus \{j\} \\ T \ni i}} p_t^{n-1} v(T) - \sum_{\substack{T \subseteq N \setminus \{i\} \\ T \ni j}} p_t^{n-1} v(T) \right]. \quad (5.5.4)
$$

For a fixed player $i \in N$, summing up (5.5.4) over all $j \in N \setminus \{i\}$ and using some straightforward combinatorial computations, we obtain

$$
(n-1)\Phi_i(N,v) = \sum_{j \in N \setminus \{i\}} \Phi_j(N,v) + \frac{\lambda}{p_1^1} \left[\sum_{\substack{S \subsetneq N \\ S \ni i}} (n-s) p_s^{n-1} v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} s p_s^{n-1} v(S) \right].
$$

Since Φ is efficient, therefore, for every game $\langle N,v \rangle$ and all $i \in N$, we arrive at the following

$$
\Phi_i(N,v) = \frac{v(N)}{n} + \frac{\lambda}{np_1^1} \left[\sum_{\substack{S \subsetneq N \\ S \ni i}} (n-s) p_s^{n-1} v(S) - \sum_{\substack{S \not\ni i \\ S \neq \emptyset}} s p_s^{n-1} v(S) \right]. \quad (5.5.5)
$$

From Corollary 2.2.8, we know that Φ is linear, symmetric and efficient.

We turn to the uniqueness part. Suppose that there exist two values Φ and ψ on $\mathcal{G}$ both possessing the λ-standardness for two-person games and $\mathcal{B}$-consistency. By considering (5.5.4) on both Φ and ψ, we derive the following equality:

$$
\Phi_i(N,v) - \Phi_j(N,v) = \psi_i(N,v) - \psi_j(N,v), \quad \text{for all } i,j \in N, i \neq j.
$$

For a fixed player i, summing up the equalities over all $j \in N \setminus \{i\}$ and using the efficiency of both values Φ and ψ, respectively, we arrive that

$$
\Phi_i(N,v) = \psi_i(N,v), \quad \text{for all } i \in N. \qquad \square
$$

By Theorem 5.4.5 and Theorem 5.5.4, both linear consistency and $\mathcal{B}$-consistency together with the λ-standardness for two-person games, characterize the class of linear, symmetric and efficient values. But the linearly reduced game and the $\mathcal{B}$-reduced game are actually different. The $\mathcal{B}$-reduced game is path-independent as shown in Proposition 5.3.5. Generally, the linearly reduced game (5.4.3) is not path-independent. In [102], Yanovskaya and Driessen showed that it is path-independent if and only if there exists a weight collection $m = \{m^n\}_{n=1}^\infty$, such that $\alpha_{t,s,r}^n = \frac{m_{s+r}^n}{m_s^{n-t}}$, for all $1 \leq t \leq n-1$, all $1 \leq s \leq n-t-1$ and all $0 \leq r \leq t$, and

$$
v_{N\setminus T}^x(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \sum\limits_{R \subseteq T} \frac{m_{s+r}^n}{m_s^{n-t}}\Big[v(S \cup R) - x(R)\Big], & \text{otherwise.} \end{cases} \tag{5.5.6}
$$

The normalized condition $\sum\limits_{r=0}^{t} \binom{t}{r}\alpha_{t,s,r}^n = 1$ is translated into $\sum\limits_{s=1}^{n-1} \binom{n-2}{s-1}m_s^n = 1$, and the relation $m_s^n = m_s^{n+1} + m_{s+1}^{n+1}$, for all n and all $1 \leq s \leq n-1$. Therefore, the path-independently linearly reduced game is the m-reduced game with a consistent weight collection m. A value Φ is said to be *path-independently linearly consistent* if Φ satisfies the reduced game property with respect to the m-reduced game with a consistent weight collection m. The $\mathcal{B}$-reduced game is introduced on the idea of $\mathcal{B}$-scaled version in terms of the relationship between any linear, symmetric and efficient value and the Shapley value. These two types of reduced games coincide on Sobolev's reduced game for the Shapley value.

In view of the p-version $\mathcal{B}$-reduced game (5.5.2), it is similar to the path-independently linearly reduced game (5.5.6) when $p_s^{n-1} = m_s^n$, for all n and all $1 \leq s \leq n-1$. Different to (5.5.2), the inverse Pascal triangle condition $m_s^n = m_s^{n+1} + m_{s+1}^{n+1}$, for all n and all $1 \leq s \leq n-1$, is needed in (5.5.6) to keep the path-independence property. Whereas, in (5.5.2), the path-independence property is kept in terms of the adaption that revise $x(R)$ by $\frac{1}{p_{s+r}^{n-1}} \cdot \frac{x(R)}{(s+r)\binom{n-1}{s+r}}$. Since, for any game $\langle N, v \rangle$ with $n \geq 3$, for all $S \subsetneq N \setminus \{i,j\}$, $S \neq \emptyset$, by (5.4.1) and (5.3.2), we have

$$\left(v_{N\setminus\{i\}}^{x}\right)_{N\setminus\{i,j\}}^{x}(S) = \frac{m_s^{n-1}}{m_s^{n-2}} v_{N\setminus\{i\}}^{x}(S) + \frac{m_{s+1}^{n-1}}{m_s^{n-2}} \left[v_{N\setminus\{i\}}^{x}(S \cup \{j\}) - x_j\right]$$

$$= \frac{m_s^{n-1}}{m_s^{n-2}} \left\{ \frac{m_s^{n}}{m_s^{n-1}} v(S) + \frac{m_{s+1}^{n}}{m_s^{n-1}} \left[v(S \cup \{i\}) - x_i\right] \right\}$$

$$+ \frac{m_{s+1}^{n-1}}{m_s^{n-2}} \left\{ \frac{m_{s+1}^{n}}{m_{s+1}^{n-1}} v(S \cup \{j\}) + \frac{m_{s+2}^{n}}{m_{s+1}^{n-1}} \left[v(S \cup \{j\} \cup \{i\}) - x_i\right] \right\} - \frac{m_{s+1}^{n-1}}{m_s^{n-2}} \cdot x_j$$

$$= \frac{m_s^{n}}{m_s^{n-2}} v(S) + \frac{m_{s+1}^{n}}{m_s^{n-2}} v(S \cup \{i\}) + \frac{m_{s+1}^{n}}{m_s^{n-2}} v(S \cup \{j\}) + \frac{m_{s+2}^{n}}{m_s^{n-2}} v(S \cup \{j,i\})$$

$$- \frac{m_{s+1}^{n} + m_{s+2}^{n}}{m_s^{n-2}} \cdot x_i - \frac{m_{s+1}^{n-1}}{m_s^{n-2}} \cdot x_j$$

$$= \frac{m_s^{n}}{m_s^{n-2}} v(S) + \frac{m_{s+1}^{n}}{m_s^{n-2}} v(S \cup \{i\}) + \frac{m_{s+1}^{n}}{m_s^{n-2}} v(S \cup \{j\}) + \frac{m_{s+2}^{n}}{m_s^{n-2}} v(S \cup \{j,i\})$$

$$- \frac{m_{s+1}^{n-1}}{m_s^{n-2}} \cdot x_i - \frac{m_{s+1}^{n-1}}{m_s^{n-2}} \cdot x_j,$$

$$\left(v_{N\setminus\{i\}}^{\mathcal{B},x}\right)_{N\setminus\{i,j\}}^{\mathcal{B},x}(S) = \frac{p_s^{n-2}}{p_s^{n-3}} v_{N\setminus\{i\}}^{\mathcal{B},x}(S) + \frac{p_{s+1}^{n-2}}{p_s^{n-3}} \left[v_{N\setminus\{i\}}^{\mathcal{B},x}(S \cup \{j\}) - \frac{1}{p_{s+1}^{n-2}} \cdot \frac{x_j}{(n-2)\binom{n-3}{s}}\right]$$

$$= \frac{p_s^{n-2}}{p_s^{n-3}} \left[\frac{p_s^{n-1}}{p_s^{n-2}} v(S) + \frac{p_{s+1}^{n-1}}{p_s^{n-2}} \left[v(S \cup \{i\}) - \frac{1}{p_{s+1}^{n-1}} \cdot \frac{x_i}{(n-1)\binom{n-2}{s}}\right]\right]$$

$$+ \frac{p_{s+1}^{n-2}}{p_s^{n-3}} \left[\frac{p_{s+1}^{n-1}}{p_{s+1}^{n-2}} v(S \cup \{j\}) + \frac{p_{s+2}^{n-1}}{p_{s+1}^{n-2}} \left[v(S \cup \{j\} \cup \{i\}) - \frac{1}{p_{s+2}^{n-1}} \cdot \frac{x_i}{(n-1)\binom{n-2}{s+1}}\right]\right]$$

$$- \frac{p_{s+1}^{n-2}}{p_s^{n-3}} \cdot \frac{1}{p_{s+1}^{n-2}} \cdot \frac{x_j}{(n-2)\binom{n-3}{s}}$$

$$= \frac{p_s^{n-1}}{p_s^{n-3}} v(S) + \frac{p_{s+1}^{n-1}}{p_s^{n-3}} v(S \cup \{i\}) + \frac{p_{s+1}^{n-1}}{p_s^{n-3}} v(S \cup \{j\}) + \frac{p_{s+2}^{n-1}}{p_s^{n-3}} v(S \cup \{j,i\})$$

$$- \frac{1}{p_s^{n-3}} \cdot \frac{x_i}{(n-2)\binom{n-3}{s}} - \frac{1}{p_s^{n-3}} \cdot \frac{x_j}{(n-2)\binom{n-3}{s}}.$$

The path-independence property is deduced from the symmetry of the above items containing either i or j.

Theorem 5.5.5. *A value Φ on $\mathcal{G}$ satisfies the standardness for two-person games and the path-independently linear consistency if and only if it is the additive efficient normalization of a semivalue.*

Proof. Let Φ be a value on $\mathcal{G}$ satisfying the standardness for two-person games and the path-independently linear consistency. Since the reduced game is path-independent, the corresponding weight collection m is consistent. By Proposition 5.4.1, we know that Φ is the additive efficient normalization of a

semivalue. On the other side, the additive efficient normalization of a semi-value is a least square value LS^m with the weight collection m such that $m_s^n = p_s^{n-1}$, for all $n \geq 1$ and all $1 \leq s \leq n - 1$. Hence, m is consistent and the linear reduced game is path-independent. By Theorem 5.4.3, LS^m satisfies the standardness for two-person games and the path-independently linear consistency. $\square$

During the proof of Theorem 5.5.4, we notice that for any linear, symmetric and efficient value, the scaling numbers of $x(R)$ of the corresponding $\mathcal{B}$-reduced game (5.5.1) or (5.5.2), can be chosen arbitrarily, except for relating the sizes of coalitions S, T and R for symmetry property. So, we introduce the $\mathcal{B}_{\mathcal{W}}$-*reduced game* as follows (p-version). Denote by $\mathcal{W}$ a collection $\{w^n\}_{n=2}^{\infty}$ of weights $w_{t,s,r}^n$ for all n and all $1 \leq t \leq n - 1$, $1 \leq s \leq n - t - 1$, $0 \leq r \leq t$.

Definition 5.5.6. *Given any game $\langle N, v \rangle$, any coalition $T \in \Omega$, $T \neq N$, any collection $\mathcal{B}$, any weight collection $\mathcal{W}$, and any payoff vector $x \in \mathbb{R}^N$, the $\mathcal{B}_{\mathcal{W}}$-reduced game $\langle N \setminus T, v_{N \setminus T}^{\mathcal{B}, x_{\mathcal{W}}} \rangle$ with respect to $\mathcal{B}$, $\mathcal{W}$, and x is defined as*

$$v_{N \setminus T}^{\mathcal{B}, x_{\mathcal{W}}}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \sum_{R \subseteq T} \dfrac{p_{s+r}^{n-1}}{p_s^{n-t-1}} \left[v(S \cup R) - w_{t,s,r}^n x(R) \right], & \text{otherwise.} \end{cases} \tag{5.5.7}$$

Theorem 5.5.7. *A linear, symmetric and efficient value Φ on $\mathcal{G}$ is $\mathcal{B}$-consistent if and only if it satisfies $\mathcal{B}_{\mathcal{W}}$-consistency, (i.e., the reduced game property with respect to the $\mathcal{B}_{\mathcal{W}}$-reduced game (5.5.7)).*

Proof. Let Φ be a linear, symmetric and efficient value on $\mathcal{G}$. For any $\mathcal{B}_{\mathcal{W}}$-reduced game $\langle T, v_T^{\mathcal{B}, x_{\mathcal{W}}} \rangle$, by Corollary 2.2.8, there exists a certain sequence of real numbers ρ_s^t, $s = 1, 2, \cdots, t - 1$, such that, for any $i \in T$,

$$\Phi_i(T, v_T^{\mathcal{B}, x_{\mathcal{W}}}) = \frac{v_T^{\mathcal{B}, x_{\mathcal{W}}}(T)}{t} + \sum_{\substack{S \subsetneq T \\ S \ni i}} \frac{\rho_s^t}{s} \cdot v_T^{\mathcal{B}, x_{\mathcal{W}}}(S) - \sum_{\substack{S \subsetneq T \\ S \not\ni i}} \frac{\rho_s^t}{t - s} \cdot v_T^{\mathcal{B}, x_{\mathcal{W}}}(S).$$

For a fixed $R \subseteq N \setminus T$, considering all coalitions with the same size s in the right hand of the above equality, the sum of these items $x(R)$ appeared in all

$v_T^{\mathcal{B},x_{\mathcal{W}}}(S)$ is

$$\sum_{\substack{|S|=s \\ S \ni i}} \frac{\rho_s^t}{s} \cdot \frac{p_{s+r}^{n-1}}{p_s^{n-t-1}} w_{t,s,r}^n x(R) - \sum_{\substack{|S|=s \\ S \not\ni i}} \frac{\rho_s^t}{t-s} \cdot \frac{p_{s+r}^{n-1}}{p_s^{n-t-1}} w_{t,s,r}^n x(R)$$

$$= \rho_s^t \cdot \frac{p_{s+r}^{n-1}}{p_s^{n-t-1}} w_{t,s,r}^n x(R) \Big[\sum_{\substack{|S|=s \\ S \ni i}} \frac{1}{s} - \sum_{\substack{|S|=s \\ S \not\ni i}} \frac{1}{t-s} \Big]$$

$$= \rho_s^t \cdot \frac{p_{s+r}^{n-1}}{p_s^{n-t-1}} w_{t,s,r}^n x(R) \Big[\binom{t-1}{s-1}\frac{1}{s} - \binom{t-1}{s}\frac{1}{t-s} \Big] = 0.$$

This implies that for $\Phi_i(T, v_T^{\mathcal{B},x_{\mathcal{W}}})$ the total sum of all payoffs $x(R)$, $R \subseteq N \setminus T$, appeared in two summation parts on the right hand of the above formula, equals 0.

For the corresponding $\mathcal{B}$-reduced game $\langle T, v_T^{\mathcal{B},x} \rangle$, we have the same conclusion. Note that $v_T^{\mathcal{B},x_{\mathcal{W}}}(T) = v_T^{\mathcal{B},x}(T)$, so we have

$$\Phi_i(T, v_T^{\mathcal{B},x_{\mathcal{W}}}) = \Phi_i(T, v_T^{\mathcal{B},x}), \quad \text{for all } i \in T.$$

Therefore, Φ is $\mathcal{B}$-consistent if and only if it is $\mathcal{B}_{\mathcal{W}}$-consistent. $\qquad \square$

Similarly, we define the following $\mathcal{W}$-linearly reduced game for any weight collection $\mathcal{W}$ and show the equivalence of $\mathcal{W}$-linear consistency and linear consistency for the corresponding linear, symmetric and efficient value.

Definition 5.5.8. *Given any game $\langle N, v \rangle$, any coalition $T \in \Omega$, $T \neq N$, any weight collection $\mathcal{W}$, and any payoff vector $x \in \mathbb{R}^N$, the $\mathcal{W}$-linearly reduced game $\langle N \setminus T, v_{N \setminus T}^{x_{\mathcal{W}}} \rangle$ with respect to $\mathcal{W}$ and x is defined as*

$$v_{N \setminus T}^{x_{\mathcal{W}}}(S) = \begin{cases} v(N) - x(T), & \text{if } S = N \setminus T; \\ \sum_{R \subseteq T} \alpha_{t,s,r}^n \big[v(S \cup R) - w_{t,s,r}^n x(R) \big], & \text{otherwise,} \end{cases} \quad (5.5.8)$$

where $\alpha_{t,s,r}^n$ are arbitrary non-negative real numbers such that $\sum_{r=0}^{t} \binom{t}{r} \alpha_{t,s,r}^n = 1$, for all $1 \leq t \leq n-1$, and all $1 \leq s \leq n-t-1$.

Theorem 5.5.9. *A linear, symmetric and efficient value Φ on $\mathcal{G}$ is linearly consistent if and only if it is $\mathcal{W}$-linearly consistent, (i.e., the reduced game property with respect to the $\mathcal{W}$-linearly reduced game (5.5.8)).*

Proof. The proof is similar to Theorem 5.5.7.   □

For any linear, symmetric and efficient value, by the equivalence of the $\mathcal{W}$-scaled version of the $\mathcal{B}$-reduced game (respectively, the linearly reduced game) and the original of reduced game, if we study the consistency for this value, we can choose some simple weighted versions, such as all $w_{t,s,r}^n = 0$ or 1. The corresponding reduced games are determined by the formulae (5.4.5) and (5.5.5) of the value.

Note that, if the collection $\mathcal{B}$ is positive and verifies the normalization condition and the inverse Pascal triangle condition (5.3.5) as given in Remark 5.3.9 (for simplicity, we call this collection $\mathcal{B}$ is *consistent*), then, the p-version of B-reduced game (5.5.2) is the adaption of the path-independently linearly reduced game (5.5.6) when $p_s^{n-1} = m_s^n$, for all n and all $1 \leq s \leq n-1$, by revised $x(R)$ as $\frac{1}{p_{s+r}^{n-1}} \cdot \frac{x(R)}{(s+r)\binom{n-1}{s+r}}$. Since the additive efficient normalization of a semivalue is linear, symmetric and efficient, by taking all $w_{t,s,r}^n = 1$ and Theorem 5.5.7, we have

Corollary 5.5.10. *The additive efficient normalization of a semivalue on $\mathcal{G}$ is $\mathcal{B}$-consistent with respect to a consistent collection $\mathcal{B}$ if and only if it is path-independently linearly consistent.*

By this corollary, together with Theorem 5.3.8 and Theorem 5.5.5, we have the following axiomatizations of the additive efficient normalization of a semivalue.

Corollary 5.5.11. *The additive efficient normalization of a semivalue is the unique value on $\mathcal{G}$ that satisfies the path-independently linear consistency, symmetry and covariance .*

Corollary 5.5.12. *Given a consistent collection $\mathcal{B}$, the additive efficient normalization of a semivalue is the unique value on $\mathcal{G}$ that satisfies the standardness for two-person games and $\mathcal{B}$-consistency.*

To conclude this chapter, we summarize by the following diagram for these two types of consistency and the corresponding axiomatizations of the class of linear, symmetric and efficient values. We denote a linear, symmetric and efficient value by LSE-value.

- LSE-value: $\mathcal{B}$-consistency $\Longleftrightarrow$ $\mathcal{B}_{\mathcal{W}}$-consistency. (Theorem 5.5.7)

- LSE-value: linear consistency $\Longleftrightarrow$ $\mathcal{W}$-linear consistency. (Theorem 5.5.9)

- ESE-value: $\mathcal{B}$-consistency with respect to a consistent collection $\mathcal{B}$ $\Longleftrightarrow$ path-independently linear consistency. (Corollary 5.5.10)

- LSE-value $\Longleftarrow$ linear consistency $+$ λ-standardness for two-person games. (Yanovskaya and Driessen [102])

- LSE-value $\Longleftrightarrow$ $\mathcal{B}$-consistency $+$ λ-standardness for two-person games. (Theorem 5.5.4)

- ESE-value $\Longleftrightarrow$ path-independently linear consistent LS-value. (Ruiz *et al.* [73])

- ESE-value $\Longleftrightarrow$ $\mathcal{B}$-consistency with respect to a consistent collection $\mathcal{B}$ $+$ symmetry $+$ covariance. (Theorem 5.3.8)

- ESE-value $\Longleftrightarrow$ path-independently linear consistency $+$ symmetry $+$ covariance. (Corollary 5.5.11)

- ESE-value $\Longleftrightarrow$ path-independently linear consistency $+$ standardness for two-person games. (Theorem 5.5.5)

- ESE-value $\Longleftrightarrow$ $\mathcal{B}$-consistency with respect to a consistent collection $\mathcal{B}$ $+$ standardness for two-person games. (Corollary 5.5.12)

Chapter 6

Matrix approach to the Harsanyi set and the Weber set

In this chapter, we apply the matrix approach to analyze the Harsanyi set and the Weber set. The Harsanyi set consists of so-called Harsanyi payoff vectors and is exactly determined by the collection P^H of dividend sharing systems. We use the matrix representation M^p for a dividend sharing system in P^H. Under the Moebius transformation, the dividend sharing matrix M^p is translated into another sharing matrix M^q, which is associated with the worth vector v of the game space $\mathcal{G}^N$. From suitable properties of the sharing matrix M^q, especially the non-member sharing entries $[M^q]_{i,S}$, $S \in \Omega, i \notin S$, we verify the axiomatization of Harsanyi payoff vectors in terms of linearity, efficiency, null player property and positivity (Derks, Haller and Peters [16]).

On the other side, the marginal contribution vectors $m^\pi(v)$, $\pi \in \Pi^N$, defined with respect to the worth system, are shown to be represented by the dividend sharing matrices M^{p^π}, $\pi \in \Pi^N$, which implies that the Weber set $W(v)$ is a subset of the Harsanyi set $H(v)$ for any game $\langle N, v \rangle$. In the setting of the Weber set, the corresponding collection $\mathcal{P}^W$ of dividend sharing matrices and the collection $\mathcal{Q}^W$ of worth sharing matrices are correlated by the inverse of the Moebius transformation. We characterize the Weber set in terms of appropriately chosen Harsanyi payoff vectors by investigating the extreme points of $\mathcal{P}^W$ as well as $\mathcal{Q}^W$. This matrix approach yields an essential

111

interpretation for the relationship between the two collections $\mathcal{P}^W$ and $\mathcal{Q}^W$, as well as the former related results of Vasil'ev [96], Vasil'ev and van der Laan [95], Derks, van der Laan and Vasil'ev [19]. Similarly, for a Harsanyi payoff vector, the complementary dividend sharing matrix $M^{\bar{p}}$ with respect to the complementary dividends is also described in terms of the complementary Moebius transformation.

The study of the extreme points of $\mathcal{P}^W$ and $\mathcal{Q}^W$ is pivotal to characterize the Weber set by the Harsanyi payoff vectors. Recall that an extreme point of a linear system $X(A, b)$ can be recognized by its carrier. A linear system associated to $\mathcal{P}^W$ and $\mathcal{Q}^W$ is constructed and the second approach to investigate their extreme points is accessed by the concept of carrier. We apply the same technique to study the extreme points of the Harsanyi set. Together with the core-type structure of the Harsanyi set by Vasil'ev [92], independently derived in Derks *et al.* [16], we present a recursive algorithm for computing the extreme points of the Harsanyi set for any game.

6.1   The Harsanyi set

With the basis of the unanimity games $\langle N, u_S \rangle$, $S \in \Omega$, for the game space $\mathcal{G}^N$, there are associated real-valued coefficients named dividends [40, 41]. As mentioned in Subsection 1.3.1 and Section 2.1, for the dividend vector Δ^v of a game $\langle N, v \rangle$, there exists the Moebius transformation M^Δ of the worth vector v such that $M^\Delta \cdot v = \Delta^v$. Equivalently, $v = (M^\Delta)^{-1} \cdot \Delta^v$, where the inverse of matrix M^Δ is given by

$$
\left[(M^\Delta)^{-1} \right]_{S,T} = \begin{cases} 1, & \text{if } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.1}
$$

That is, $v(S) = \sum\limits_{T \subseteq S} \Delta^v(T)$ for all $S \in \Omega$. In particular, $v(N) = \sum\limits_{T \in \Omega} \Delta^v(T)$. In words, the worth of a coalition is the sum of dividends of all subcoalitions.

For the reason that the square-coalitional matrix M^Δ is nonsingular, the dividend vector Δ^v and the worth vector v are equivalent from the algebraic point of view. So, a payoff vector allocating the overall worth $v(N)$ among players, in a sense, distributes these dividends $\Delta^v(S)$, $S \in \Omega$, among players. This procedure of distributing dividends elicits some other game-theoretic solution concepts (*e.g.*, the weighted Shapley values). The Harsanyi set, consisting of all Harsanyi payoff vectors, is well-known.

A *dividend sharing system* $p = (p_i^S)_{i \in S}^{S \in \Omega}$ is defined as

$$
p_i^S \geq 0 \text{ for all } S \in \Omega, i \in S, \quad \text{such that } \sum_{i \in S} p_i^S = 1 \text{ for every } S \in \Omega.
$$

For any game $\langle N, v \rangle$ and any dividend sharing system p, the dividend sharing payoff vector $\varphi^p(v) \in \mathbb{R}^N$ given by

$$
\varphi_i^p(v) = \sum_{S \ni i} p_i^S \Delta^v(S), \quad \text{for all } i \in N,
$$

is called the *Harsanyi payoff vector* or *sharing value* with respect to p. According to the sharing value, the payoff $\varphi_i^p(v)$ to player i equals the sum over all coalitions S containing i of the dividend sharings p_i^S by player i in the dividend $\Delta^v(S)$ of coalition S. It is well known that the Shapley value $Sh(v)$ is the sharing value of symmetrically distributed dividends, *i.e.*,

$$
Sh_i(v) = \sum_{S \ni i} \frac{\Delta^v(S)}{s}, \quad \text{for all } i \in N.
$$

The set of Harsanyi payoff vectors agrees with the set of all payoff vectors obtained by distributing the dividends of all coalitions S among the players in S. This set is named the *Harsanyi set* by Vasil'ev [91, 93], or the *selectope* by Hammer, Peled and Sorensen [39], independently. Denote by P^H the set of all dividend sharing systems p, *i.e.*,

$$P^H = \{p \mid p_i^S \geq 0, \sum_{j \in S} p_j^S = 1, \text{ for all } S \in \Omega, i \in S\}.$$

Formally, the *Harsanyi mapping* is the solution mapping that assigns, to each game $\langle N, v \rangle$, the *Harsanyi set* $H(v) \subseteq \mathbb{R}^N$ given by

$$H(v) = \{\varphi^p(v) \mid p \in P^H\}.$$

6.2 The Moebius transformation for characterizing the Harsanyi payoff vectors

In this section, the matrix approach is applied to express and characterize Harsanyi payoff vectors and the Harsanyi set. For any dividend sharing system, the corresponding Harsanyi payoff vector is linear with respect to the game space. By Theorem 2.2.1, the dividend sharing system can be modelled as a matrix, which distributes, for any coalition, its dividend among its members.

The *dividend sharing matrix* $M^p = \begin{bmatrix} M^p \end{bmatrix}_{i \in N, S \in \Omega}$ with respect to a dividend sharing system $p \in P^H$ is the column-coalitional matrix given by

$$\begin{bmatrix} M^p \end{bmatrix}_{i,S} = \begin{cases} p_i^S, & \text{if } i \in S; \\ 0, & \text{if } i \notin S. \end{cases}$$

Since $[M^p]_{i,S} = 0$ for all $i \in S$, the condition $\sum_{i \in S} p_i^S = 1$ means that each column sum of M^p equals one, *i.e.*, $\sum_{i \in N} [M^p]_{i,S} = 1$ for all $S \in \Omega$, or equivalently,

$$\mathbf{1}_N' \cdot M^p = \mathbf{1}_\Omega', \tag{6.2.1}$$

where $\mathbf{1}_N$ and $\mathbf{1}_\Omega$ are column vectors with all entries equal to one. Therefore, the set P^H of dividend sharing systems may be identified with its matrix equivalence

$$\mathcal{P}^H = \{M^p \mid M^p \geq \mathbf{0}; \mathbf{1}_N' \cdot M^p = \mathbf{1}_\Omega'; \begin{bmatrix} M^p \end{bmatrix}_{i,S} = 0 \text{ for all } S \in \Omega, i \notin S\}. \tag{6.2.2}$$

For a game $\langle N, v \rangle$, with respect to a dividend sharing matrix $M^p \in \mathcal{P}^H$, the matrix representation of any Harsanyi payoff vector $\varphi^p(v) = M^p \cdot \Delta^v$ holds, and hence, the matrix equivalence of the Harsanyi set is given by

$$H(v) = \{M^p \cdot \Delta^v \mid M^p \in \mathcal{P}^H\}.$$

For sharing the dividends, the Harsanyi set is strongly related to the basis of unanimity games. By Theorem 2.1.8, the Moebius operator transforms this basis into the standard basis of unity games. It means that the Moebius transformation translates any dividend sharing matrix into another type of sharing matrix associated with the worth vector instead of the dividend vector.

Let the column-coalitional matrix $M^q = [M^q]_{i \in N, S \in \Omega}$, denote the right Moebius transformation $M^q = M^p M^\Delta$ of any dividend sharing matrix M^p, and call M^q the *worth sharing matrix*. We denote by $\mathcal{Q}^H$ the set of all worth sharing matrices, *i.e.*, the right Moebius transformation on the set $\mathcal{P}^H$ of dividend sharing matrices:

$$\mathcal{Q}^H = \mathcal{P}^H M^\Delta = \{M^p M^\Delta \mid M^p \in \mathcal{P}^H\}. \tag{6.2.3}$$

The following result presents the characterization of worth sharing matrices.

Theorem 6.2.1. *The set $\mathcal{Q}^H$ of all worth sharing matrices is given by*

$$\mathcal{Q}^H = \Big\{ M^q \mid \mathbf{1}'_N \cdot M^q = (0, 0, \cdots, 0, 1);$$
$$[M^q]_{i,S} = -[M^q]_{i,S \setminus \{i\}}, \text{ for all } S \in \Omega, i \in S, S \neq \{i\};$$
$$\sum_{T \supseteq S} [M^q]_{i,T} \geq 0, \text{ for all } S \in \Omega, i \in S \Big\}. \tag{6.2.4}$$

Proof. Let $M^p \in \mathcal{P}^H$ and write $M^q = M^p M^\Delta$. We show that M^q satisfies these three conditions. By (6.2.1) and Proposition 2.1.6, we have

$$\mathbf{1}'_N \cdot M^q = \mathbf{1}'_N \cdot M^p M^\Delta = \mathbf{1}'_\Omega \cdot M^\Delta = (0, 0, \cdots, 0, 1).$$

By (2.1.2), for all $i \in N$, $S \in \Omega$,

$$[M^q]_{i,S} = \sum_{T \in \Omega} [M^p]_{i,T} [M^\Delta]_{T,S} = \sum_{T \supseteq S} (-1)^{t-s} [M^p]_{i,T}.$$

Recall $[M^p]_{i,T} = 0$, for all $T \in \Omega$, $i \notin T$. So, for all $S \in \Omega$, $i \in S, S \neq \{i\}$, it follows that

$$[M^q]_{i,S\setminus\{i\}} = \sum_{T \supseteq S\setminus\{i\}} (-1)^{t-s+1}[M^p]_{i,T} = -\sum_{T \supseteq S}(-1)^{t-s}[M^p]_{i,T} = -[M^q]_{i,S}.$$

Further, as $M^p = M^q(M^\Delta)^{-1}$, we obtain from (6.1.1) that

$$0 \leq [M^p]_{i,S} = \sum_{T \in \Omega}[M^q]_{i,T}\big[(M^\Delta)^{-1}\big]_{T,S} = \sum_{T \supseteq S}[M^q]_{i,T}, \quad \text{for all } S \in \Omega, i \in S.$$

Hence, $M^q \in \mathcal{Q}^H$. Now, suppose that M^{q^*} satisfies these three conditions. Write $M^{p^*} = M^{q^*}(M^\Delta)^{-1}$. We show that $M^{p^*} \in \mathcal{P}^H$. By (6.1.1), we have

$$\mathbf{1}'_N \cdot M^{p^*} = \mathbf{1}'_N \cdot M^{q^*}(M^\Delta)^{-1} = (0, 0, \cdots, 0, 1) \cdot (M^\Delta)^{-1} = \mathbf{1}'_\Omega.$$

For all $S \in \Omega$ and all $i \in N$, we have

$$[M^{p^*}]_{i,S} = \sum_{T \in \Omega}[M^{q^*}]_{i,T}\big[(M^\Delta)^{-1}\big]_{T,S} = \sum_{T \supseteq S}[M^{q^*}]_{i,T}.$$

If $S \ni i$, then $[M^{p^*}]_{i,S} \geq 0$. If $S \not\ni i$, then

$$\begin{aligned}
[M^{p^*}]_{i,S} &= \sum_{T \supseteq S}[M^{q^*}]_{i,T} = \sum_{\substack{T \supseteq S \\ T \ni i}}[M^{q^*}]_{i,T} + \sum_{\substack{T \supseteq S \\ T \not\ni i}}[M^{q^*}]_{i,T} \\
&= \sum_{\substack{T \supseteq S \\ T \ni i}}[M^{q^*}]_{i,T} + \sum_{\substack{T \supseteq S \\ T \ni i}}[M^{q^*}]_{i,T\setminus\{i\}} \\
&= \sum_{\substack{T \supseteq S \\ T \ni i}}[M^{q^*}]_{i,T} - \sum_{\substack{T \supseteq S \\ T \ni i}}[M^{q^*}]_{i,T} = 0.
\end{aligned}$$

We conclude that $M^{p^*} \in \mathcal{P}^H$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By (2.1.1) and (6.2.3), a Harsanyi payoff vector $\varphi^p(v)$ of any game $\langle N, v\rangle$ corresponding to the dividend sharing matrix $M^p \in \mathcal{P}^H$ can be expressed as

$$\varphi^p(v) = M^p \cdot \Delta^v = M^p M^\Delta \cdot v = M^q \cdot v, \quad \text{where } M^q \in \mathcal{Q}^H.$$

In words, any sharing value given by a weighted sum of dividends can be rewritten by the Moebius transformation matrix as a weighted sum of worths, and vice versa. Some related results can also be found in Vasil'ev [94] and Dragan [21]. The following is an alternative representation for the Harsanyi set in terms of the worth system of any game.

Corollary 6.2.2. *For every game* $\langle N, v \rangle$, *its Harsanyi set is of the form*

$$H(v) = \{M^q \cdot v \mid M^q \in \mathcal{Q}^H\}.$$

Hence, the Harsanyi set $H(v)$ is described algebraically in terms of the worth sharing matrices in $\mathcal{Q}^H$, modelled as a second, alternative type of matrices. The structure of these matrices, particularly the occurrence of non-member entries $[M^q]_{i,T \setminus \{i\}}$ for all $T \in \Omega, T \ni i, T \neq \{i\}$, reflects the properties of efficiency, null player property, and positivity, which are used to characterize Harsanyi payoff vectors (Derks, Haller, and Peters [16]).

It is easy to check that Harsanyi payoff vectors satisfy linearity (additivity), efficiency, dummy player property (null player property), inessential game property and covariance. A game $\langle N, v \rangle$ is said to be *totally positive* (respectively, *negative*), if its dividends $\Delta^v(S) \geq 0$ (respectively, $\Delta^v(S) \leq 0$) for all $S \in \Omega$. In this setting, the positivity of a value Φ on $\mathcal{G}^N$ is introduced as follows (see Vasil'ev [90]).

- *Positivity*: a value Φ on $\mathcal{G}^N$ is positive if $\Phi_i(v) \geq 0$ for every totally positive game $\langle N, v \rangle$, all $i \in N$, as well as $\Phi_i(v) \leq 0$ for every totally negative game $\langle N, v \rangle$, all $i \in N$.

Derks, Haller, and Peters [16] characterized the Harsanyi payoff vectors through the positivity property.

Theorem 6.2.3 (Derks *et al.* [16]). *A value* Φ *on* $\mathcal{G}^N$ *satisfies linearity, efficiency, the null player property and positivity if and only if* Φ *is a Harsanyi payoff vector.*

Proof by the matrix approach. For any game $\langle N, v \rangle$, let $\varphi^p(v)$ be a Harsanyi payoff vector with respect to the dividend sharing matrix $M^p \in \mathcal{P}^H$. Then $\varphi^p(v) = M^q \cdot v$, where $M^q = M^p M^\Delta$ and $M^q \in \mathcal{Q}^H$.

By Theorem 2.2.1, we know that $\varphi^p(v)$ is linear. In viewing of (6.2.4), by Proposition 2.2.2, the condition $\mathbf{1}'_N \cdot M^q = (0, 0, \cdots, 0, 1)$ of M^q implies that $\varphi^p(v)$ is efficient; whereas $[M^q]_{i,S} = -[M^q]_{i,S \setminus \{i\}}$ for all $i \in S$, $S \in \Omega$, $S \neq \{i\}$, by Proposition 2.2.3, implies the null player property of $\varphi^p(v)$. Since $\varphi^p(v) = M^q \cdot v = M^p \cdot \Delta^v$, and $[M^p]_{i,S} = \sum_{T \supseteq S} [M^q]_{i,T} \geq 0$ for all $i \in S$, $S \in \Omega$, for every totally positive game, we have

$$\varphi^p_i(v) = \sum_{S \ni i} [M^p]_{i,S} \Delta^v(S) \geq 0, \quad \text{for all } i \in N.$$

The positivity holds for φ^p.

Let Φ be a value on $\mathcal{G}^N$ satisfying these four properties. We show that Φ is a Harsanyi payoff vector. By Theorem 2.2.1, we know that there exists a column-coalitional matrix M^Φ such that $\Phi(v) = M^\Phi \cdot v$ for any game $\langle N, v \rangle$. Together with efficiency, we have $\mathbf{1}'_N \cdot M^\Phi = (0, 0, \cdots, 0, 1)$, from Proposition 2.2.2. For all $S \in \Omega$ and $i \in S, S \neq \{i\}$, $[M^\Phi]_{i,S} = -[M^\Phi]_{i,S \setminus \{i\}}$ is deduced from the null player property and Proposition 2.2.3. Moreover, by Corollary 2.2.4 we have

$$\Phi_i(v) = \sum_{S \ni i} [M^\Phi]_{i,S} \big[v(S) - v(S \setminus \{i\})\big], \quad \text{for all } i \in N.$$

Consider the unanimity game $\langle N, u_S \rangle$, for every $S \in \Omega$. It is a totally positive game with $\Delta^{u_S}(S) = 1$ and $\Delta^{u_S}(T) = 0$, for all $T \neq S, T \in \Omega$. By the null player property, we have $\Phi_i(u_S) = 0$, for all $i \notin S$. For all $i \in S$, it holds that

$$\Phi_i(u_S) = \sum_{T \ni i} [M^\Phi]_{i,T} \big(u_S(T) - u_S(T \setminus \{i\})\big) = \sum_{T \supseteq S} [M^\Phi]_{i,T}.$$

So for all $S \in \Omega, i \in S$, by positivity we have $\sum_{T \supseteq S} [M^\Phi]_{i,T} \geq 0$. Therefore, $M^\Phi \in \mathcal{Q}^H$. By Corollary 6.2.2, we have $\Phi(v) \in H(v)$. $\qquad\square$

Because the Harsanyi set is defined on the dividend vector instead of the traditional worth vector, we would like to unveil the meaning of this solution on the latter system. In this algebraic proof, the sharing matrix M^p of dividends is translated by the Moebius transformation into the associated sharing matrix M^q of worths.

Viewing the proof of Theorem 6.2.3, the worth sharing system with respect to a Harsanyi payoff vector φ^p, modelled as the column-coalitional matrix M^q, implies the linearity of φ^p. The conditions: $\mathbf{1}'_N \cdot M^p = (0, 0, \cdots, 0, 1)$, $[M^q]_{i,S} = -[M^q]_{i,S \setminus \{i\}}$, for all $S \in \Omega, i \in S, S \neq \{i\}$, and $\sum_{T \supseteq S} [M^q]_{i,T} \geq 0$, for all $S \in \Omega, i \in S$, are corresponding to the properties of efficiency, the null player property and positivity of the Harsanyi payoff vector, respectively. Correspondingly, by the proof of Theorem 6.2.1, the dividend sharing matrix M^p, implies the linearity of φ^p. The conditions: $\mathbf{1}'_N \cdot M^p = \mathbf{1}'_\Omega$, $[M^p]_{i,S} = 0$, for all $S \in \Omega, i \notin S$, and $[M^p]_{i,S} \geq \mathbf{0}$, for all $i \in N, S \in \Omega$, are also

corresponding to the properties of efficiency, the null player property and positivity of ϕ^p, respectively. Particularly, the Shapley value is the Harsanyi payoff vector possessing symmetry. By Proposition 2.2.5, the corresponding dividend sharing matrix M^p is also symmetric, that is $[M^p]_{i,S} = \frac{1}{s}$ for all $S \in \Omega, i \in S$.

Remark 6.2.4. *In* [95], *Vasil'ev and van der Laan introduced the collection Q_1^* of all sharing systems $q = (q_i^S)_{i \in S}^{S \in \Omega}$ associated with the Harsanyi set, which assign a weight $q_i^S \in \mathbb{R}$ of $v(S)$ to each player i in any coalition $S \ni i$. The sharing systems q are characterized by two equalities:*

$$q(N) = \sum_{i \in N} q_i^N = 1, \quad and \quad \sum_{i \in S} q_i^S = \sum_{j \in N \setminus S} q_j^{S \cup \{j\}}, \quad for\ all\ S \in \Omega.$$

It is rather difficult to understand the relationship of the second equality with axioms of the Harsanyi payoff vectors. In our matrix approach, the non-member entries $[M^q]_{i,S \setminus \{i\}}, S \in \Omega, i \in S, S \neq \{i\}$, are also considered. During the proof of Theorem 6.2.3, *we know that the equalities*

$$[M^q]_{i,S} = -[M^q]_{i,S \setminus \{i\}}, \ S \neq \{i\}, \quad \sum_{T \supseteq S} [M^q]_{i,T} \geq 0, \ for\ all\ S \in \Omega, i \in S,$$

imply the null player property and positivity, respectively. So by the matrix approach, some essential properties of the associated worth sharing matrices $M^q \in \mathcal{Q}^H$ are presented for axiomatizing the Harsanyi payoff vector as well as the Harsanyi set. Moreover, it motivates a shorter and intuitive proof of the characterization of the Weber set by the Harsanyi payoff vectors in Section 6.3.

6.3 Matrix approach to characterize the Weber set

The Weber set is based on the concept of marginal contribution vectors, as the the collection of all random order values. In a sense, it is to share the worths of a game. By the inverse of the Moebius transformation, we can represent the marginal contribution vectors as a sum of appropriate dividends to discuss the relationship between a random order value and a sharing value, as well as the relationship between the Weber set and the Harsanyi set.

Derks, Haller and Peters [16] have shown that the Weber set is a subset of the Harsanyi set and they also provide necessary and sufficient conditions on a

game $\langle N, v \rangle$ for the Harsanyi set to coincide with the Weber set. Vasil'ev and van der Laan [95] characterized the Weber set by Harsanyi payoff vectors. In the following, we prove these results by using the matrix approach. Firstly, the marginal contribution vectors $m^\pi(v)$, $\pi \in \Pi^N$, are represented algebraically in terms of the dividend sharing matrix $M^p \in \mathcal{P}^H$.

Definition 6.3.1. *Given any game $\langle N, v \rangle$ and any permutation $\pi \in \Pi^N$, the marginal contribution vector $m^\pi(v)$ of (1.3.15) is represented by the matrix M^{q^π} as*

$$m^\pi(v) = M^{q^\pi} \cdot v,$$

where the matrix $M^{q^\pi} = [M^{q^\pi}]_{i \in N, S \in \Omega}$ is column-coalitional defined by

$$[M^{q^\pi}]_{i,S} = \begin{cases} 1, & \text{if } S = \pi^i; \\ -1, & \text{if } S = \pi^i \setminus \{i\}, \pi^i \neq \{i\}; \\ 0, & \text{otherwise.} \end{cases} \qquad (6.3.1)$$

By the interrelationship between the worth vector and the dividend vector for a game, we have the following expression for the marginal contribution vectors.

Lemma 6.3.2. *For any game $\langle N, v \rangle$ and permutation $\pi \in \Pi^N$, the marginal contribution vector $m^\pi(v)$ is of the form $m^\pi(v) = M^{p^\pi} \cdot \Delta^v$, where the matrix $M^{p^\pi} = [M^{p^\pi}]_{i \in N, S \in \Omega}$ is column-coalitional given by $M^{p^\pi} = M^{q^\pi}(M^\Delta)^{-1}$, in explicit form*

$$[M^{p^\pi}]_{i,S} = \begin{cases} 1, & \text{if } i \in S, \ S \subseteq \pi^i; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, $M^{p^\pi} \in \mathcal{P}^H$ for all $\pi \in \Pi^N$.

Proof. For any game $\langle N, v \rangle$ and any permutation $\pi \in \Pi^N$, by Definition 6.3.1 and Definition 2.1.4, we have

$$m^\pi(v) = M^{q^\pi} \cdot v = M^{q^\pi}(M^\Delta)^{-1} \cdot \Delta^v = M^{p^\pi} \cdot \Delta^v, \text{ where } M^{p^\pi} = M^{q^\pi}(M^\Delta)^{-1}.$$

By (6.3.1) and (6.1.1), for all $i \in N$, $S \in \Omega$,

$$[M^{p^\pi}]_{i,S} = \sum_{T \in \Omega} [M^{q^\pi}]_{i,T} [(M^\Delta)^{-1}]_{T,S} = \sum_{T \supseteq S} [M^{q^\pi}]_{i,T}.$$

Observing (6.3.1), there are at most two non-zero entries in each row of M^{q^π}. Since $S \subseteq \pi^i \setminus \{i\}$ implies $S \subseteq \pi^i$, we have $[M^{p^\pi}]_{i,S} \geq 0$ for all $i \in N$, $S \in \Omega$.

Moreover, $[M^{p^\pi}]_{i,S} = 1$ if and only if $S \subseteq \pi^i$ and $S \not\subseteq \pi^i \setminus \{i\}$. That is to say, $S \subseteq \pi^i$ with $i \in S$. Otherwise, we have $[M^{p^\pi}]_{i,S} = 0$, and particularly, $[M^{p^\pi}]_{i,S} = 0$, for all $S \in \Omega$, $i \notin S$. Finally, it is easy to check that $\mathbf{1}'_N \cdot M^{p^\pi} = \mathbf{1}'_\Omega$. Therefore, $M^{p^\pi} \in \mathcal{P}^H$. $\qquad\square$

By this lemma, the Weber set can be represented in terms of the worth or dividend sharing matrices, associated with permutations, as

$$W(v) = \text{Conv}\{M^{q^\pi} \cdot v \mid \pi \in \Pi^N\} = \text{Conv}\{M^{p^\pi} \cdot \Delta^v \mid \pi \in \Pi^N\}.$$

From the fact that $M^{p^\pi} \in \mathcal{P}^H$ for all $\pi \in \Pi^N$, it is deduced that, for any game $\langle N, v \rangle$, all marginal contribution vectors $m^\pi(v)$, $\pi \in \Pi^N$, belong to the Harsanyi set $H(v)$. Since the Harsanyi set is convex, it includes the Weber set.

Corollary 6.3.3 (Derks *et al.* [16], Vasil'ev and van der Laan, [95]). *It holds that $C(v) \subseteq W(v) \subseteq H(v)$, for all games $\langle N, v \rangle$.*

In Derks *et al.* [16] the class of games is characterized for which the two sets $W(v)$ and $H(v)$ coincide by using one type of consistency property, see also Derks and Peters [17]. In the following, we focus on a characterization of the dividend sharing matrices for which the corresponding Harsanyi payoff vectors belong to the Weber set. This characterization has first been discussed in Vasil'ev and van der Laan [95], later in Derks, van der Laan and Vasil'ev [19]. We provide here an alternative and more accessible approach, which is from a matrix-theoretic point of view related to the inverse of the Moebius transformation.

We introduce the set $\mathcal{P}^W$ of dividend sharing matrices as follows:

$$\mathcal{P}^W = \{M^p \in \mathcal{P}^H \mid \left[M^p M^\Delta\right]_{i,S} \geq 0 \text{ for all } S \in \Omega, i \in S\}.$$

Correspondingly, we denote by $\mathcal{Q}^W$ the right Moebius transformation of $\mathcal{P}^W$, *i.e.*,

$$\mathcal{Q}^W = \mathcal{P}^W M^\Delta = \{M^p M^\Delta \mid M^p \in \mathcal{P}^W\}. \tag{6.3.2}$$

We show that both of these sets are related to the Weber set. The following characterization is a direct consequence of (6.2.3).

Proposition 6.3.4.

$$\mathcal{Q}^W = \{M^q \in \mathcal{Q}^H \mid [M^q]_{i,S} \geq 0 \text{ for all } S \in \Omega, i \in S\}.$$

From this proposition, for any $M^q \in \mathcal{Q}^W$, since

$$\left[M^p\right]_{i,S} = \left[M^q(M^\Delta)^{-1}\right]_{i,S} = \sum_{T \in \Omega}[M^q]_{i,T}\left[(M^\Delta)^{-1}\right]_{T,S} = \sum_{T \supseteq S}[M^q]_{i,T},$$

and $[M^q]_{i,T} \geq 0$ for all $i \in T$, we have $[M^p]_{i,R} \geq [M^p]_{i,S}$ for all $R \subseteq S, S \in \Omega$, with $i \in R$. We call this *monotonicity* of the dividend sharing matrix $M^p \in \mathcal{P}^W$.

Obviously, $\mathcal{P}^W$ is a convex polyhedron. For every $M^p \in \mathcal{P}^W$, it holds that $0 \leq [M^p]_{i,S} \leq 1$, for all $i \in N, S \in \Omega$, and so, $\mathcal{P}^W$ is a bounded convex set, as is $\mathcal{Q}^W$. Since any bounded convex polyhedron is equal to the convex hull of its extreme points, we turn to study the extreme points of $\mathcal{P}^W$ and $\mathcal{Q}^W$. Let $Ex(Y)$ denote the set of all extreme points of a convex set Y.

Lemma 6.3.5 (Vasil'ev and van der Laan [95]). *The extreme points of the polyhedron $\mathcal{P}^W$ are the matrices M^{p^π}, $\pi \in \Pi^N$, that is*

$$Ex(\mathcal{P}^W) = \{M^{p^\pi} \mid \pi \in \Pi^N\} \quad and \quad \mathcal{P}^W = Conv\{M^{p^\pi} \mid \pi \in \Pi^N\}.$$

Proof. Firstly, we show that $M^{p^\pi} \in Ex(\mathcal{P}^W)$, for all $\pi \in \Pi^N$. By Lemma 6.3.2, $M^{p^\pi} \in \mathcal{P}^H$ and, for all $S \in \Omega$, $i \in S$, we have

$$\left[M^{p^\pi}M^\Delta\right]_{i,S} = [M^{q^\pi}]_{i,S} \geq 0.$$

So, $M^{p^\pi} \in \mathcal{P}^W$. Suppose that there exist $M^{p^1}, M^{p^2} \in \mathcal{P}^W$ and $\lambda \in (0,1)$ such that

$$M^{p^\pi} = \lambda M^{p^1} + (1 - \lambda)M^{p^2}.$$

Note that $[M^{p^\pi}]_{i,S} \in \{0,1\}$, while $0 \leq [M^{p^1}]_{i,S} \leq 1$, $0 \leq [M^{p^2}]_{i,S} \leq 1$, for all $i \in N, S \in \Omega$. Thus, if $[M^{p^\pi}]_{i,S} = 0$, then $[M^{p^1}]_{i,S} = [M^{p^2}]_{i,S} = 0$; else if $[M^{p^\pi}]_{i,S} = 1$, then $[M^{p^1}]_{i,S} = [M^{p^2}]_{i,S} = 1$. Therefore, $M^{p^\pi} = \lambda M^{p^1} + (1 - \lambda)M^{p^2}$, together with $0 < \lambda < 1$, imply $M^{p^1} = M^{p^2} = M^{p^\pi}$. Hence, $M^{p^\pi} \in Ex(\mathcal{P}^W)$ and $\{M^{p^\pi} \mid \pi \in \Pi^N\} \subseteq Ex(\mathcal{P}^W)$.

Now we prove that $Ex(\mathcal{P}^W) \subseteq \{M^{p^\pi} \mid \pi \in \Pi^N\}$. Let M^{p^*} be an extreme point of $\mathcal{P}^W$ and $M^{q^*} = M^{p^*}M^\Delta$. We show that $[M^{p^*}]_{i,S} \in \{0,1\}$, for all $i \in N, S \in \Omega$.

Suppose that S_1 is the last column in the lexicographic order, such that $0 < [M^{p^*}]_{i_1,S_1} < 1$, for some $i_1 \in S_1$. So $[M^{p^*}]_{i_1,R} = [M^{p^*}]_{j_1,R} = 0$, for all $R \supsetneq S_1$. Obviously, $|S_1| \geq 2$, for the reason of $[M^{p^*}]_{i_1,i_1} = 1$. There exists $j_1 \in S_1$ such that $0 < [M^{p^*}]_{j_1,S_1} < 1$ because of $\mathbf{1}'_N \cdot M^{p^*} = \mathbf{1}'_\Omega$. Therefore,

$$[M^{q^*}]_{i_1,S_1} = \left[M^{p^*} M^\Delta\right]_{i_1,S_1} = \sum_{R \supseteq S_1} (-1)^{r-s_1} [M^p]_{i_1,R} = [M^{p^*}]_{i_1,S_1} > 0,$$

and similarly, $[M^{q^*}]_{j_1,S_1} = [M^{p^*}]_{j_1,S_1} > 0$.

So $[M^{q^*}]_{i_1,S_1\setminus\{i_1\}} = -[M^{q^*}]_{i_1,S_1} < 0$ and $[M^{q^*}]_{j_1,S_1\setminus\{j_1\}} = -[M^{q^*}]_{j_1,S_1} < 0$. Together with $\mathbf{1}'_N \cdot M^{q^*} = (0, 0, \cdots, 0, 1)$, it follows that there exists $i_2 \in S_2 = S_1 \setminus \{i_1\}$, $j_2 \in T_2 = S_1 \setminus \{j_1\}$ such that $[M^{q^*}]_{i_2,S_2} > 0$ and $[M^{q^*}]_{j_2,T_2} > 0$. This procedure can be continued up to coalitions with a single player. We get two sequences $i_1, i_2, \cdots, i_{|S_1|}$ and $j_1, j_2, \cdots, j_{|S_1|}$ in S_1 satisfying

$$[M^{q^*}]_{i_l,S_l} > 0 \text{ and } [M^{q^*}]_{j_l,T_l} > 0, \quad \text{for } l = 1, 2, \cdots, |S_1|,$$

where $S_1 = T_1$, $S_{l+1} = S_l \setminus \{i_l\}$ and $T_{l+1} = T_l \setminus \{j_l\}$, $l = 1, 2, \cdots, |S_1| - 1$. Some of them maybe repeated between these two sequences. Let

$$a = \min\{[M^{q^*}]_{i_l,S_l}, [M^{q^*}]_{j_l,T_l} \mid l = 1, 2, \cdots, |S_1|\}.$$

Obviously, $a > 0$. Construct two sharing systems M^{q^a} and $M^{q^{-a}}$ as following

$$
\begin{cases}
\left[M^{q^a}\right]_{i_l,S_l} = [M^{q^*}]_{i_l,S_l} + a, \ l = 1, 2, \cdots, |S_1|; \\
\left[M^{q^a}\right]_{i_l,S_{l+1}} = [M^{q^*}]_{i_l,S_{l+1}} - a, \ l = 1, 2, \cdots, |S_1| - 1; \\
\left[M^{q^a}\right]_{j_l,T_l} = [M^{q^*}]_{j_l,T_l} - a, \ l = 1, 2, \cdots, |S_1|; \\
\left[M^{q^a}\right]_{j_l,T_{l+1}} = [M^{q^*}]_{j_l,T_{l+1}} + a, \ l = 1, 2, \cdots, |S_1| - 1; \\
\left[M^{q^a}\right]_{i,S} = [M^{q^*}]_{i,S}, \quad \text{otherwise,}
\end{cases}
$$

$$
\begin{cases}
\left[M^{q^{-a}}\right]_{i_l,S_l} = [M^{q^*}]_{i_l,S_l} - a, \ l = 1, 2, \cdots, |S_1|; \\
\left[M^{q^{-a}}\right]_{i_l,S_{l+1}} = [M^{q^*}]_{i_l,S_{l+1}} + a, \ l = 1, 2, \cdots, |S_1| - 1; \\
\left[M^{q^{-a}}\right]_{j_l,T_l} = [M^{q^*}]_{j_l,T_l} + a, \ l = 1, 2, \cdots, |S_1|; \\
\left[M^{q^{-a}}\right]_{j_l,T_{l+1}} = [M^{q^*}]_{j_l,T_{l+1}} - a, \ l = 1, 2, \cdots, |S_1| - 1; \\
\left[M^{q^{-a}}\right]_{i,S} = [M^{q^*}]_{i,S}, \quad \text{otherwise.}
\end{cases}
$$

There are possibly some repetitions between the two sequences of $\left[M^{q^a}\right]_{i_l,S_l}$ and $\left[M^{q^a}\right]_{j_l,T_l}$ for $l = 1, 2, \cdots, |S_1|$ or between the two sequences of $\left[M^{q^a}\right]_{i_l,S_{l+1}}$ and $\left[M^{q^a}\right]_{j_l,T_{l+1}}$ for $l = 1, 2, \cdots, |S_1| - 1$. If this happens, the repetition entries will not change by adding both a and $-a$. A similar statement applies to $M^{q^{-a}}$.

Obviously, $M^{q^*} = \frac{1}{2}(M^{q^a} + M^{q^{-a}})$. Furthermore, we have $\mathbf{1}'_N \cdot M^{q^a} = \mathbf{1}'_N \cdot M^{q^{-a}} = (0, 0, \cdots, 0, 1)$, $\left[M^{q^{-a}}\right]_{i,S} \geq 0$, $[M^{q^a}]_{i,S} \geq 0$, for all $S \in \Omega$, $i \in S$, and also, for $S \neq \{i\}$, $\left[M^{q^a}\right]_{i,S\setminus\{i\}} = -\left[M^{q^a}\right]_{i,S}, [M^{q^{-a}}]_{i,S\setminus\{i\}} = -[M^{q^{-a}}]_{i,S}$. Therefore,

$$M^{q^a} \in \mathcal{Q}^W, \text{and } M^{q^{-a}} \in \mathcal{Q}^W.$$

So by (6.3.2), $M^{p^a} = M^{q^a}(M^\Delta)^{-1} \in \mathcal{P}^W$ and $M^{p^{-a}} = M^{q^{-a}}(M^\Delta)^{-1} \in \mathcal{P}^W$. Thus

$$M^{p^*} = M^{q^*}(M^\Delta)^{-1} = \frac{1}{2}(M^{q^a} + M^{q^{-a}})(M^\Delta)^{-1} = \frac{1}{2}(M^{p^a} + M^{p^{-a}}).$$

This contradicts that M^{p^*} is an extreme point. And hence $[M^{p^*}]_{i,S} \in \{0, 1\}$, for all $i \in N, S \in \Omega$.

It remains to show that the extreme point M^{p^*}, is a M^{p^π} for some $\pi \in \Pi^N$. By $[M^{p^*}]_{i,S} \in \{0, 1\}$, for all $i \in N, S \in \Omega$ and $\mathbf{1}'_N \cdot M^{p^*} = \mathbf{1}'_\Omega$, it holds that there exists one and only one entry $[M^{p^*}]_{i,S} = 1$ in each column of M^{p^*} indexed by any $S \in \Omega$. Let $[M^{p^*}]_{k_1,N} = 1$ for the column of M^{p^*} indexed by N. Furthermore, by monotonicity of M^{p^*}, we have $[M^{p^*}]_{k_1,S} = 1$, for all $S \subseteq N$, $S \ni k_1$. Considering the column indexed by $N \setminus \{k_1\}$, let $[M^{p^*}]_{k_2,N\setminus\{k_1\}} = 1$. By monotonicity we know that $[M^{p^*}]_{k_2,S} = 1$, for all $S \subseteq N \setminus \{k_1\}$, $S \ni k_2$. Continuing this procedure, we obtain a sequence $k_1, k_2, \cdots, k_n$ satisfying

$$[M^{p^*}]_{k_r,N\setminus\{k_1,k_2,\cdots,k_{r-1}\}} = 1, \quad \text{for } r = 2, 3, \cdots, n,$$

and

$$[M^{p^*}]_{k_r,S} = 1, \quad \text{for all } S \ni k_r, S \subseteq N \setminus \{k_1, k_2, \cdots, k_{r-1}\},$$

and $[M^{p^*}]_{i,T} = 0$ for other cases. It is easy to check that $M^{p^*} = M^{p^\pi}$ with $\pi = (k_n, k_{n-1}, \cdots, k_1) \in \Pi^N$. This implies that $Ex(\mathcal{P}^W) \subseteq \{M^{p^\pi} \mid \pi \in \Pi^N\}$. $\quad\square$

By the relationship between $\mathcal{P}^W$ and $\mathcal{Q}^W$, we have the following corollary.

Corollary 6.3.6 (Vasil'ev and van der Laan [95], Vasil'ev [96]). *The extreme points of the polyhedron $\mathcal{Q}^W$ are the matrices M^{q^π}, $\pi \in \Pi^N$, that is*

$$Ex(\mathcal{Q}^W) = \{M^{q^\pi} \mid \pi \in \Pi^N\} \quad and \quad \mathcal{Q}^W = Conv\{M^{q^\pi} \mid \pi \in \Pi^N\}.$$

Proof. Since $\mathcal{Q}^W = \{M^p M^\Delta \mid M^p \in \mathcal{P}^W\} = \mathcal{P}^W M^\Delta$, it is an invertible linear transformation from $\mathcal{P}^W$ to $\mathcal{Q}^W$, or equivalently, a one-one mapping from $\mathcal{P}^W$ to $\mathcal{Q}^W$. Both $\mathcal{Q}^W$ and $\mathcal{P}^W$ are convex bounded sets. Hence

$$Ex(\mathcal{Q}^W) = Ex(\mathcal{P}^W)M^\Delta = \{M^{p^\pi}M^\Delta \mid \pi \in \Pi^N\} = \{M^{q^\pi} \mid \pi \in \Pi^N\}.$$

That is to say, $\mathcal{Q}^W = Conv\{M^{q^\pi} \mid \pi \in \Pi^N\}$. $\qquad\square$

Theorem 6.3.7. *For every game $\langle N, v \rangle$, the Weber set $W(v)$ is the subset of the Harsanyi set $H(v)$ given by*

$$W(v) = \{M^q \cdot v \mid M^q \in \mathcal{Q}^W\}.$$

Proof. It follows immediately from

$$W(v) = Conv\{m^\pi(v) \mid \pi \in \Pi^N\} = Conv\{M^{q^\pi} \cdot v \mid \pi \in \Pi^N\}$$

and Corollary 6.3.6 concerning $Ex(\mathcal{Q}^W) = \{M^{q^\pi} \mid \pi \in \Pi^N\}$. $\qquad\square$

From this theorem, we deduce the characterization of random order values by Weber [99] and the characterization of the Weber set in terms of Harsanyi payoff vectors by Vasil'ev and van der Laan [95].

Theorem 6.3.8 (Weber [99]). *A value Φ on $\mathcal{G}^N$ satisfies linearity, efficiency, the null player property and monotonicity if and only if Φ is a random order value.*

Proof by the matrix approach. It is easy to check that a random order value satisfies linearity, efficiency, the null player property and monotonicity.

We only prove that a value possessing these four properties is a random order value. Similar to the proof of Theorem 6.2.3, we know that a linear, efficient value Φ satisfies the null player property if and only if there exists a column-coalitional matrix M^Φ with conditions: $\mathbf{1}'_N \cdot M^\Phi = (0, 0, \cdots, 0, 1)$,

and $[M^\Phi]_{i,S} = -[M^\Phi]_{i,S\setminus\{i\}}$, for all $S \in \Omega$, $i \in S, S \neq \{i\}$, such that for any game $\langle N, v \rangle$,

$$\Phi_i(v) = \sum_{S \ni i} [M^\Phi]_{i,S} \big[v(S) - v(S \setminus \{i\})\big], \quad \text{for all } i \in N.$$

By using the monotonicity of Φ on the unity game $\langle N, e^S \rangle$, $S \in \Omega$, it holds that, for all $i \in S$,

$$\Phi_i(e^S) = \sum_{T \ni i} [M^\Phi]_{i,T} \big[e^S(T) - e^S(T \setminus \{i\})\big] = [M^\Phi]_{i,S} \geq 0.$$

Therefore, $M^\Phi \in \mathcal{Q}^W$. By Theorem 6.3.7, Φ is a random order value.   □

Theorem 6.3.9 (Vasil'ev and van der Laan [95]). *For every game $\langle N, v \rangle$, the Weber set $W(v)$ is the subset of the Harsanyi set $H(v)$ given by*

$$W(v) = \{M^p \cdot \Delta^v \mid M^p \in \mathcal{P}^W\}.$$

Proof. By Theorem 6.3.7, (6.3.2) and (2.1.1), we have

$$W(v) = \{M^q \cdot v \mid M^q \in \mathcal{Q}^W\} = \{M^p M^\Delta \cdot v \mid M^p \in \mathcal{P}^W\}$$
$$= \{M^p \cdot \Delta^v \mid M^p \in \mathcal{P}^W\}.   \quad\Box$$

Remark 6.3.10. *For these characterizations of both the random order values and the Weber set by the Harsanyi payoff vectors above, the most pivotal procedure is to analyze the extreme points of $\mathcal{P}^W$ and $\mathcal{Q}^W$. It was first shown in the work of Vasil'ev [96] that the matrices M^{q^π} are the extreme points of $\mathcal{Q}^W$. Then, Vasil'ev and van der Laan [95] presented an alternative and shorter approach, but it needed several lemmata filling 9 pages with rather difficult proofs. In our approach, we prefer to study $\mathcal{P}^W$ instead of $\mathcal{Q}^W$ for the dividends to be directly related to the Harsanyi payoff vectors. As we mentioned in Remark 6.2.4, our description of properties for $\mathcal{Q}^W$ deduces a shorter and intuitive characterizing procedure. Comparing to the former approaches, our matrix expression yields an essential interpretation for the relation between two sharing systems $\mathcal{P}^W$ and $\mathcal{Q}^W$, as well as some works of Vasil'ev in [96], Vasil'ev and van der Laan in [95], Derks, van der Laan, and Vasil'ev in [19].*

Derks *et al.* [16] showed that the dividend sharing system p is monotonic if the Harsanyi payoff vector $\varphi^p(v)$ of a game $\langle N, v \rangle$ is in the Weber set. However,

not every $\varphi^p(v)$ is in the Weber set when p is monotonic. To characterize the monotonic dividend sharing systems yielding Harsanyi payoff vectors in the Weber set, Derks, van der Laan and Vasil'ev introduced in [19] the property named strong monotonicity for a dividend sharing system p. Similarly, in terms of the matrix representation, a dividend sharing matrix $M^p \in \mathcal{P}^H$ is called *strongly monotonic* if

$$\sum_{R|S \subseteq R \subseteq T} (-1)^{r-s}[M^p]_{i,R} \geq 0, \quad \text{whenerver } i \in S \subseteq T \subseteq N.$$

This property characterized the dividend sharing matrix $M^p \in \mathcal{P}^W$ associated the Weber set.

Theorem 6.3.11 (Derks *et al.* [19]). *A dividend sharing matrix $M^p \in \mathcal{P}^H$ is strongly monotonic if and only if $M^p \in \mathcal{P}^W$.*

This theorem provides us the second characterization by Derks *et al.* [19] of the random order values within the Harsanyi set.

Theorem 6.3.12 (Derks *et al.* [19]). *For each game $\langle N, v \rangle$, the Harsanyi payoff vector $x \in H(v)$ belongs to the Weber set $W(v)$ if and only if there exists a strong monotonic dividend sharing matrix $M^p \in \mathcal{P}^H$ such that $x = \varphi^p(v)$.*

6.4 The complementary dividend sharing matrices

As known, there is associated a coefficient vector with each basis for the game space, such as the worth vector of the basis of unity games and the dividend vector of the basis of unanimity games. These coefficient vectors are equivalent from the algebraic point of view. As shown in Section 2.2 and in this chapter, any linear value on the game space can be viewed as how to share the worth vector, equivalently, how to share the coefficient vectors related to different bases. The sharing system can be formulated algebraically as a column-coalitional matrix.

In this section, the complementary dividends with reference to the basis of complementary unanimity games are discussed and the corresponding complementary dividend sharing matrices are analyzed.

For any game $\langle N, v \rangle$, its dividend vector Δ^v and complementary dividend vector $\bar{\Delta}^v$ are represented by the Moebius transformation matrix M^Δ and

the complementary Moebius transformation matrix $M^{\bar{\Delta}}$ respectively, as $\Delta^v = M^{\Delta} \cdot v$ and $\quad \bar{\Delta}^v = M^{\bar{\Delta}} \cdot v$ (see Definitions 2.1.2 and 2.1.4). On the one hand, $v = (M^{\bar{\Delta}})^{-1} \cdot \bar{\Delta}^v$, on the other, by (1.3.8), the game representation $v = \sum_{S \subsetneq N} \bar{\Delta}^v(S) \bar{u}_S$ yields that

$$v(T) = \sum_{S \subsetneq N} \bar{\Delta}^v(S) \bar{u}_S(T) = \sum_{T \subseteq N \setminus S} \bar{\Delta}^v(S), \quad \text{for all } T \in \Omega.$$

Proposition 6.4.1. *The inverse* $(M^{\bar{\Delta}})^{-1} = \left[(M^{\bar{\Delta}})^{-1}\right]_{T \in \Omega, S \subsetneq N}$ *of matrix* $M^{\bar{\Delta}}$ *is given by*

$$\left[(M^{\bar{\Delta}})^{-1}\right]_{T,S} = \begin{cases} 1, & \text{if } T \subseteq N \setminus S; \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.1}$$

In accordance with Remark 2.1.5, the last column of the inverse $(M^{\bar{\Delta}})^{-1}$ is indexed by the empty set $\emptyset$ such that $\left[(M^{\bar{\Delta}})^{-1}\right]_{T,\emptyset} = 1$ for all $T \in \Omega$. Since the last row of $(M^{\bar{\Delta}})^{-1}$ is indexed by the grand coalition N, by (6.4.1), each entry equals zero, except for $\left[(M^{\bar{\Delta}})^{-1}\right]_{N,\emptyset} = 1$. For future purpose, we restate this property as

$$(0, 0, \cdots, 0, 1) \cdot (M^{\bar{\Delta}})^{-1} = (0, 0, \cdots, 0, 1). \tag{6.4.2}$$

We have the following relationship between these two types of dividend systems.

Proposition 6.4.2. *For any game* $\langle N, v \rangle$, *it holds that* $\bar{\Delta}^v = A \cdot \Delta^v$, *where the transformation matrix* $A = \left[A\right]_{T \subsetneq N, S \in \Omega}$ *is square-coalitional given by* $A = M^{\bar{\Delta}}(M^{\Delta})^{-1}$, *in explicit form*

$$[A]_{T,S} = \begin{cases} (-1)^t, & \text{if } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.3}$$

Proof. Obviously, for any game $\langle N, v \rangle$, we have

$$\bar{\Delta}^v = M^{\bar{\Delta}} \cdot v = M^{\bar{\Delta}}(M^{\Delta})^{-1} \cdot \Delta^v = A \cdot \Delta^v, \quad \text{where } A = M^{\bar{\Delta}}(M^{\Delta})^{-1}.$$

For all $T \subsetneq N, S \in \Omega$, write $Q = (N \setminus T) \cup S$. In view of (6.1.1) and (2.1.6),

we have

$$[A]_{T,S} = \sum_{R \in \Omega} \left[M^{\bar{\Delta}}\right]_{T,R}\left[(M^{\Delta})^{-1}\right]_{R,S} = \sum_{\substack{N \setminus R \subseteq T \\ R \supseteq S}} (-1)^{t-(n-r)}$$

$$= \sum_{\substack{R \supseteq N \setminus T \\ R \supseteq S}} (-1)^{t+r-n} = \sum_{R \supseteq [(N \setminus T) \cup S]} (-1)^{t+r-n}$$

$$= \sum_{R \supseteq Q} (-1)^{t+r-n} = \sum_{r=q}^{n} (-1)^{t+r-n} \binom{n-q}{r-q}$$

$$= (-1)^{t+q-n} \sum_{k=0}^{n-q} (-1)^k \binom{n-q}{k} = (-1)^{t+q-n}(1-1)^{n-q}$$

$$= \begin{cases} (-1)^t, & \text{if } Q = N, \text{ i.e., } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Note that the last row of A is indexed by the empty set $\emptyset$ such that every entry of this row is equal to one. That is to say,

$$(0, 0, \cdots, 0, 1) \cdot A = \mathbf{1}'_{\Omega}. \tag{6.4.4}$$

Proposition 6.4.3. *The inverse $A^{-1} = \left[A^{-1}\right]_{T \in \Omega, S \subsetneq N}$ of matrix A is given by*

$$\left[A^{-1}\right]_{T,S} = \begin{cases} (-1)^{t-1}, & \text{if } T \not\subseteq S; \\ 0, & \text{otherwise,} \end{cases} \tag{6.4.5}$$

And, $\mathbf{1}'_{\Omega} \cdot A^{-1} = (0, 0, \cdots, 0, 1)$.

Proof. Obviously, $A^{-1} = M^{\Delta}(M^{\bar{\Delta}})^{-1}$. By (2.1.2) and (6.4.1), for all $T \in \Omega, S \subsetneq N$, we have

$$[A^{-1}]_{T,S} = \sum_{R \in \Omega} \left[M^{\Delta}\right]_{T,R}\left[(M^{\bar{\Delta}})^{-1}\right]_{R,S} = \sum_{\substack{R \subseteq T, R \neq \emptyset \\ R \subseteq N \setminus S}} (-1)^{t-r} = \sum_{\substack{R \subseteq T \setminus S \\ R \neq \emptyset}} (-1)^{t-r}.$$

Write $Q = T \setminus S$. Then we have

$$[A^{-1}]_{T,S} = \sum_{\substack{R \subseteq Q \\ R \neq \emptyset}} (-1)^{t-r} = \sum_{r=1}^{q} (-1)^{t-r} \binom{q}{r} = (-1)^t \sum_{r=0}^{q} (-1)^r \binom{q}{r} - (-1)^t$$

$$= \begin{cases} (-1)^{t-1}, & \text{if } q \neq 0, \text{ i.e., } T \not\subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 2.1.6 and (6.4.2), it follows that

$$\mathbf{1}'_\Omega \cdot A^{-1} = \mathbf{1}'_\Omega \cdot M^\Delta (M^{\bar\Delta})^{-1} = (0,0,\cdots,0,1) \cdot (M^{\bar\Delta})^{-1} = (0,0,\cdots,0,1). \quad \square$$

For a game $\langle N,v\rangle$ and any dividend sharing matrix M^p, from Proposition 6.4.2, the Harsanyi payoff vector $\varphi^p(v)$ can also be thought as to distribute the complementary dividends $\bar\Delta^v(S)$, $S \subsetneq N$. Correspondingly, we introduce the column-coalitional matrix $M^{\bar p} = [M^{\bar p}]_{i\in N, S\subsetneq N}$, by the linear transformation $M^{\bar p} = M^p A^{-1}$ of any dividend sharing matrix M^p, and call $M^{\bar p}$ the *complementary dividend sharing matrix*. We denote by $\bar{\mathcal P}^H$ the set of all complementary dividend sharing matrices, *i.e.*,

$$\bar{\mathcal P}^H = \mathcal P^H A^{-1} = \{ M^p A^{-1} \mid M^p \in \mathcal P^H \}. \tag{6.4.6}$$

So, the Harsanyi payoff vector $\varphi^p(v)$ and the Harsanyi set $H(v)$ are given by

$$\varphi^p(v) = M^p \cdot \Delta^v = M^{\bar p} \cdot \bar\Delta^v, \quad H(v) = \{ M^{\bar p} \cdot \bar\Delta^v \mid M^{\bar p} \in \bar{\mathcal P}^H \}.$$

Note that the last column of $M^{\bar p}$ is indexed by the empty set $\emptyset$ instead of the grand coalition N. In the following, we study the properties of the complementary dividend sharing matrix $M^{\bar p} \in \bar{\mathcal P}^H$.

Theorem 6.4.4. *The set $\bar{\mathcal P}^H$ of all complementary dividend sharing matrices is given by*

$$\bar{\mathcal P}^H = \Big\{ M^{\bar p} \mid \mathbf{1}'_N \cdot M^{\bar p} = (0,0,\cdots,0,1),$$
$$[M^{\bar p}]_{i,S} = [M^{\bar p}]_{i,\emptyset}, \quad \text{for all } S \subsetneq N, i \notin S,$$
$$[M^{\bar p}]_{i,\{i\}} = [M^{\bar p}]_{i,\emptyset} - 1,$$
$$\sum_{\substack{T\subseteq S \\ T\ni i}} (-1)^t [M^{\bar p}]_{i,T} \geq 0, \text{ for all } S \subsetneq N, i \in S, S \neq \{i\} \Big\}. \tag{6.4.7}$$

Proof. Let $M^p \in \mathcal P^H$ be a dividend sharing matrix and write $M^{\bar p} = M^p A^{-1}$. We show that the complementary dividend sharing matrix $M^{\bar p}$ satisfies these four conditions. By the characterization of the dividend sharing matrices in (6.2.2) and Proposition 6.4.5, we have

$$\mathbf{1}'_N \cdot M^{\bar p} = \mathbf{1}'_N \cdot M^p A^{-1} = \mathbf{1}'_\Omega \cdot A^{-1} = (0,0,\cdots,0,1).$$

Further, for all $i \in N$, $S \subsetneq N$, by (6.4.3), it holds that

$$[M^p]_{i,S} = [M^{\bar{p}} A]_{i,S} = \sum_{\substack{T \subseteq N \\ \neq}} [M^{\bar{p}}]_{i,T} [A]_{T,S} = \sum_{T \subseteq S} (-1)^t [M^{\bar{p}}]_{i,T}. \qquad (6.4.8)$$

We show inductively on $s = |S|$ that $[M^{\bar{p}}]_{i,S} = [M^{\bar{p}}]_{i,\emptyset}$, for all $S \not\ni i$. Since $[M^p]_{i,S} = 0$ for all $S \in \Omega$, $i \notin S$. Obviously, for any $j \in N, j \neq i$, $[M^p]_{i,\{j\}} = [M^{\bar{p}}]_{i,\emptyset} - [M^{\bar{p}}]_{i,\{j\}}$, it follows that $[M^p]_{i,\{j\}} = 0$ if and only if $[M^{\bar{p}}]_{i,\{j\}} = [M^{\bar{p}}]_{i,\emptyset}$.

Assume that $[M^{\bar{p}}]_{i,T} = [M^{\bar{p}}]_{i,\emptyset}$ for all $T \in \Omega, i \notin T$ with $|T| \leq |S|$. Consider the coalition $S \cup \{l\}$, for any $l \in N \setminus S, l \neq i$. By (6.4.8) and induction hypothesis, we have

$$\begin{aligned}
[M^p]_{i,S \cup \{l\}} &= \sum_{T \subseteq S \cup \{l\}} (-1)^t [M^{\bar{p}}]_{i,T} \\
&= \sum_{T \subseteq S} (-1)^t [M^{\bar{p}}]_{i,T} + \sum_{\substack{T \subseteq S \cup \{l\} \\ T \ni l}} (-1)^t [M^{\bar{p}}]_{i,T} \\
&= [M^p]_{i,S} + \sum_{\substack{T \subsetneq S \cup \{l\} \\ T \ni l}} (-1)^t [M^{\bar{p}}]_{i,T} + (-1)^{s+1} [M^{\bar{p}}]_{i,S \cup \{l\}} \\
&= 0 + [M^{\bar{p}}]_{i,\emptyset} \sum_{t=1}^{s} (-1)^t \binom{s}{t-1} + (-1)^{s+1} [M^{\bar{p}}]_{i,S \cup \{l\}} \\
&= (-1)^s [M^{\bar{p}}]_{i,\emptyset} + (-1)^{s+1} [M^{\bar{p}}]_{i,S \cup \{l\}}.
\end{aligned}$$

Since $[M^p]_{i,S \cup \{l\}} = 0$, we conclude that $[M^{\bar{p}}]_{i,S \cup \{l\}} = [M^{\bar{p}}]_{i,\emptyset}$.

Note that $1 = [M^p]_{i,\{i\}} = [M^{\bar{p}}]_{i,\emptyset} - [M^{\bar{p}}]_{i,\{i\}}$, and therefore, $[M^{\bar{p}}]_{i,\{i\}} = [M^{\bar{p}}]_{i,\emptyset} - 1$. For all $S \subsetneq N$, $i \in S, S \neq \{i\}$,

$$\begin{aligned}
0 \leq [M^p]_{i,S} &= \sum_{T \subseteq S} (-1)^t [M^{\bar{p}}]_{i,T} = \sum_{\substack{T \subseteq S \\ T \ni i}} (-1)^t [M^{\bar{p}}]_{i,T} + \sum_{\substack{T \subseteq S \\ T \not\ni i}} (-1)^t [M^{\bar{p}}]_{i,T} \\
&= \sum_{\substack{T \subseteq S \\ T \ni i}} (-1)^t [M^{\bar{p}}]_{i,T} + \sum_{T \subseteq S \setminus \{i\}} (-1)^t [M^{\bar{p}}]_{i,T} \\
&= \sum_{\substack{T \subseteq S \\ T \ni i}} (-1)^t [M^{\bar{p}}]_{i,T} + [M^p]_{i,S \setminus \{i\}} \\
&= \sum_{\substack{T \subseteq S \\ T \ni i}} (-1)^t [M^{\bar{p}}]_{i,T}. \qquad (6.4.9)
\end{aligned}$$

Now, suppose that $M^{\bar{p}^*}$ satisfies these four conditions. $M^{\bar{p}^*}$ is a complementary dividend sharing matrix if and only if $M^{\bar{p}^*}A$ is a dividend sharing matrix. Write $M^{p^*} = M^{\bar{p}^*}A$. We show that M^{p^*} satisfies the three conditions in (6.2.2). If $\mathbf{1}'_N \cdot M^{\bar{p}^*} = (0,0,\cdots,0,1)$, then by (6.4.4),

$$\mathbf{1}'_N \cdot M^{p^*} = \mathbf{1}'_N \cdot M^{\bar{p}^*}A = (0,0,\cdots,0,1) \cdot A = (0,0,\cdots,0,1).$$

For all $i \in N, S \in \Omega$, if $S \not\ni i$, by $[M^{\bar{p}^*}]_{i,S} = [M^{\bar{p}^*}]_{i,\emptyset}$ and (6.4.8), it follows that

$$[M^{p^*}]_{i,S} = \sum_{T \subseteq S}(-1)^t [M^{\bar{p}^*}]_{i,T} = [M^{\bar{p}^*}]_{i,\emptyset}\sum_{T \subseteq S}(-1)^t$$

$$= [M^{\bar{p}^*}]_{i,\emptyset}\sum_{t=0}^{s}(-1)^t \binom{s}{t} = [M^{\bar{p}^*}]_{i,\emptyset}(1-1)^s = 0.$$

Moreover, $[M^{p^*}]_{i,\{i\}} = [M^{\bar{p}^*}]_{i,\emptyset} - [M^{\bar{p}^*}]_{i,\{i\}} = 1$, and if $i \in S, S \neq \{i\}$, applying the same procedure in (6.4.9), we conclude that

$$[M^{p^*}]_{i,S} = \sum_{\substack{T \subseteq S \\ T \ni i}}(-1)^t[M^{\bar{p}}]_{i,T} + [M^p]_{i,S\setminus\{i\}} = \sum_{\substack{T \subseteq S \\ T \ni i}}(-1)^t[M^{\bar{p}}]_{i,T} \geq 0. \qquad \square$$

In fact, the complementary dividend sharing system, modelled as a column-coalitional matrix, implies the linearity of the Harsanyi payoff vector. Recalling the characterization of the Harsanyi payoff vector by the matrix approach in Theorem 6.2.3 and the proof of Theorem 6.4.4, we know that the conditions of $M^{\bar{p}}$ in (6.4.7) are corresponding to the axioms of efficiency, the null player property and positivity of the Harsanyi payoff vector, respectively. The efficiency condition $\mathbf{1}'_N \cdot M^{\bar{p}} = (0,0,\cdots,0,1)$ can be replaced by

$$\sum_{i \in N}[M^{\bar{p}}]_{i,\emptyset} = 1, \quad \sum_{i \in S}[M^{\bar{p}}]_{i,S} = \sum_{i \in S}[M^{\bar{p}}]_{i,\emptyset} - 1, \quad \text{for all } S \neq \emptyset. \qquad (6.4.10)$$

By (6.2.3) and (6.4.6), it is easy to derive the following relationship.

Corollary 6.4.5. *The set $\mathcal{Q}^H$ of worth sharing system is the right complementary Moebius transformation of the set $\bar{\mathcal{P}}^H$, i.e.,*

$$\mathcal{Q}^H = \bar{\mathcal{P}}^H M^{\bar{\Delta}} = \{M^{\bar{p}}M^{\bar{\Delta}} \mid M^{\bar{p}} \in \bar{\mathcal{P}}^H\}. \qquad (6.4.11)$$

Remark 6.4.6. *The Shapley value is the Harsanyi payoff vector possessing symmetry. The symmetry property of the corresponding complementary dividend sharing matrix $M^{\bar{p}}$, by Proposition 2.2.5, together with (6.4.10), deduce that $[M^{\bar{p}}]_{i,\emptyset} = \frac{1}{n}$, as well as $[M^{\bar{p}}]_{i,S} = \frac{1}{n}$, for all $i \notin S$, $S \subsetneq N$. And $[M^{\bar{p}}]_{i,S} = \frac{1}{n} - \frac{1}{s}$, for all $i \in S$, $S \subsetneq N$. From the proof of Theorem 2.2.1, we know that in this case $[M^{\bar{p}}]_{i,S} = Sh_i(\bar{u}_S)$ is the Shapley value of the player i of the complementary unanimity game $\langle N, \bar{u}_S \rangle$, for all $S \subsetneq N$, as given in [29].*

Similarly, we denote by $\bar{\mathcal{P}}^W$ the linear transformation of $\mathcal{P}^W$ by A^{-1}, *i.e.*,

$$\bar{\mathcal{P}}^W = \mathcal{P}^W A^{-1} = \{ M^p A^{-1} \mid M^p \in \mathcal{P}^W \}. \tag{6.4.12}$$

So, the Weber set $W(v)$ of any game $\langle N, v \rangle$ is given by

$$W(v) = \{ M^{\bar{p}} \cdot \bar{\Delta}^v \mid M^{\bar{p}} \in \bar{\mathcal{P}}^W \}.$$

Since a Harsanyi payoff vector is included in the Weber set if and only if positivity is strengthened as monotonicity. Equivalently, by Proposition 6.3.4, its worth sharing matrix M^q satisfies $[M^q]_{i,S} \geq 0$, for all $i \in S$, $S \in \Omega$. For the complementary dividend sharing matrices in $\bar{\mathcal{P}}^W$, we have the following proposition.

Theorem 6.4.7. *The set $\bar{\mathcal{P}}^W$ of the complementary dividend sharing matrices $M^{\bar{p}}$ with respect to the Weber set, is given by*

$$\bar{\mathcal{P}}^W = \Big\{ M^{\bar{p}} \in \bar{\mathcal{P}}^H \mid [M^{\bar{p}}]_{i,\emptyset} \geq 0, and$$
$$\sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} \geq 0, for\ all\ S \ni i, S \neq \{i\} \Big\}.$$

Proof. Let $M^{\bar{p}} \in \bar{\mathcal{P}}^H$ be a complementary dividend sharing matrix. Write $M^q = M^{\bar{p}} M^{\bar{\Delta}}$ and by (6.4.11), we have $M^q \in Q^H$. Then by (6.3.2) and (6.4.12), $M^{\bar{p}} \in \bar{\mathcal{P}}^W$ if and only if $M^q \in Q^W$. Equivalently, by Proposition 6.3.4, if and only if $[M^q]_{i,S} \geq 0$, for all $S \in \Omega, i \in S$. By Theorem 6.4.4 and

(2.1.6), we have, for all $S \in \Omega, i \in S$,

$$[M^q]_{i,S} = \sum_{\substack{T \subsetneq N}} [M^{\bar{p}}]_{i,T} [M^{\bar{\Delta}}]_{T,S} = \sum_{\substack{T \supseteq N \setminus S \\ T \neq N}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T}$$

$$= \sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} + \sum_{\substack{T \supseteq N \setminus S \\ T \not\ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T}$$

$$= \sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} + [M^{\bar{p}}]_{i,\emptyset} \sum_{\substack{T \supseteq N \setminus S \\ T \not\ni i}} (-1)^{t-(n-s)}$$

$$= \sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} + [M^{\bar{p}}]_{i,\emptyset} \sum_{t=n-s}^{n-1} (-1)^{t-(n-s)} \binom{s-1}{t-(n-s)}$$

$$= \sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} + [M^{\bar{p}}]_{i,\emptyset} \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k}$$

$$= \sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} + [M^{\bar{p}}]_{i,\emptyset} (1-1)^{s-1}.$$

So for all $i \in N$,

$$[M^q]_{i,\{i\}} = \sum_{\substack{T \supseteq N \setminus \{i\} \\ T \neq N}} (-1)^{t-(n-1)} [M^{\bar{p}}]_{i,T} = [M^{\bar{p}}]_{i,N \setminus \{i\}} = [M^{\bar{p}}]_{i,\emptyset},$$

and all for $S \ni i$, $S \neq \{i\}$,

$$[M^q]_{i,S} = \sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T}.$$

Therefore, $[M^q]_{i,S} \geq 0$ for all $S \in \Omega, i \in S$ if and only if $[M^{\bar{p}}]_{i,\emptyset} \geq 0$ for all $i \in N$ and, $\sum_{\substack{T \supseteq N \setminus S \\ T \neq N, T \ni i}} (-1)^{t-(n-s)} [M^{\bar{p}}]_{i,T} \geq 0$ for all $S \ni i, S \neq \{i\}$. $\qquad \square$

At last, we present the properties of the complementary dividend sharing matrices with reference to the marginal contribution vectors.

Proposition 6.4.8. *For any game $\langle N, v \rangle$ and any permutation $\pi \in \Pi^N$, the marginal contribution vector $m^\pi(v)$ be of the form $m^\pi(v) = M^{\bar{p}^\pi} \cdot \bar{\Delta}^v$, where*

the complementary dividend sharing matrix $M^{\bar{p}^\pi}$ is given by

$$[M^{\bar{p}^\pi}]_{i,S} = \begin{cases} 1, & \text{if } \pi^i = \{i\}, \ S \subseteq N \setminus \pi^i; \\ -1, & \text{if } \pi^i \neq \{i\}, \ S \subseteq N \setminus (\pi^i \setminus \{i\}), \ S \ni i; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For any game $\langle N, v \rangle$ and any permutation $\pi \in \Pi^N$, by Definition 6.3.1 and Definition 2.1.4, we have

$$m^\pi(v) = M^{q^\pi} \cdot v = M^{q^\pi}(M^{\bar{\Delta}})^{-1} \cdot \bar{\Delta}^v = M^{\bar{p}^\pi} \cdot \bar{\Delta}^v, \text{ where } M^{\bar{p}^\pi} = M^{q^\pi}(M^{\bar{\Delta}})^{-1}.$$

By (6.3.1) and (6.4.1), for all $i \in N$, $S \subsetneq N$,

$$[M^{\bar{p}^\pi}]_{i,S} = \sum_{T \in \Omega} [M^{q^\pi}]_{i,T} [(M^{\bar{\Delta}})^{-1}]_{T,S} = \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} [M^{q^\pi}]_{i,T}.$$

Observing (6.3.1), there are at most two non-zero entries ($[M^{q^\pi}]_{i,\pi^i} = 1$ and $[M^{q^\pi}]_{i,\pi^i \setminus \{i\}} = -1$) in each row of M^{q^π}. So $[M^{\bar{p}^\pi}]_{i,S} = 1$ if and only if

$$\pi^i \subseteq N \setminus S, \text{ and } \pi^i \setminus \{i\} \not\subseteq N \setminus S \text{ or } \pi^i \setminus \{i\} = \emptyset.$$

That is to say, $S = N \setminus \pi^i, \pi^i = \{i\}$. $[M^{\bar{p}^\pi}]_{i,S} = -1$ if and only if

$$\pi^i \setminus \{i\} \subseteq N \setminus S, \ \pi^i \setminus \{i\} \neq \emptyset, \text{ and } \pi^i \not\subseteq N \setminus S.$$

That is to say, $\pi^i \neq \{i\}$, $S \subseteq N \setminus (\pi^i \setminus \{i\})$, $S \ni i$. For the other cases, we have $[M^{\bar{p}^\pi}]_{i,S} = 0$. $\qquad \square$

6.5   The extreme points of the Harsanyi set

A *selector* is a function $\Lambda : \Omega \to N$ with $\Lambda(S) \in S$ for every coalition $S \in \Omega$. The set of all selectors on Ω is denoted by Λ^N. Obviously, the total number of elements in Λ^N is $\prod_{s=1}^{n} s^{\binom{n}{s}}$, denoted by m. For every selector $\Lambda \in \Lambda^N$, we denote by M^{p^Λ} a dividend sharing matrix such that $[M^{p^\Lambda}]_{\Lambda(S),S} = 1$ and $[M^{p^\Lambda}]_{j,S} = 0$, for all $S \in \Omega$ and all $j \neq \Lambda(S)$. Clearly, the set $\mathcal{P}^H$ of all dividend sharing matrices is a convex set. Therefore, $\mathcal{P}^H$ is determined uniquely by its extreme points. For every dividend sharing matrix M^p, we denote by $\widetilde{M^p}$ the vector of all entries $[M^p]_{i,S}$, $S \in \Omega, i \in N$, indexed in lexicographic order

firstly by $S \in \Omega$ and secondly by $i \in N$. Let $\widetilde{\mathcal{P}^H}$ be the set of all vectors $\widetilde{M^p}$, $M^p \in \mathcal{P}^H$. Then $\widetilde{\mathcal{P}^H}$ can be viewed as the solution set of the linear system:

$$\begin{cases} \sum\limits_{i \in S} \left[M^p \right]_{i,S} = 1, & \text{for all } S \in \Omega; \\ \sum\limits_{i \notin S} \left[M^p \right]_{i,S} = 0, & \text{for all } S \in \Omega; \\ \left[M^p \right]_{i,S} \geq 0, & \text{for all } S \in \Omega, i \in N. \end{cases}$$

Obviously, M^p is an extreme point of $\mathcal{P}^H$ if and only if the corresponding vector $\widetilde{M^p}$ is an extreme point of $\widetilde{\mathcal{P}^H}$. For a linear system $X(A,b) = \{x \in \mathbb{R}^m_+ \mid Ax = b\}$, where $b \in \mathbb{R}^n$ and A is a $n \times m$ matrix, it is known that an extreme point of $X(A,b)$ can be recognized by its carrier. The *carrier* $K(x)$ of a vector x is defined by

$$K(x) = \{j \mid x_j \neq 0\}.$$

And one can show the following

Lemma 6.5.1 (Derks [18]). *An element x of $X(A,b)$ is an extreme point if and only if its carrier is minimal, i.e., there is no $y \in X(A,b), y \neq x$, such that $K(y) \subseteq K(x)$.*

From this lemma, we derive the next proposition concerning the extreme points of $\mathcal{P}^H$, denoted by $Ex(\mathcal{P}^H)$.

Proposition 6.5.2. *For the set $\mathcal{P}^H$ of all dividend sharing matrices, we have*

$$Ex(\mathcal{P}^H) = \{ M^{p^\Lambda} \mid \Lambda \in \Lambda^N \}.$$

Proof. By (6.2.4), for any $M^p \in \mathcal{P}^H$, it holds that $\mathbf{1}'_N \cdot M^p = \mathbf{1}'_\Omega$. So, for every $S \in \Omega$, there exists at least one player $i \in S$ satisfying $[M^p]_{i,S} > 0$. We choose one of these players i to be the selector of coalition S, i.e., $\Lambda(S) = i$. Then we have $K(\widetilde{M^{p^\Lambda}}) \subseteq K(\widetilde{M^p})$. By Lemma 6.5.1, only $\widetilde{M^{p^\Lambda}}$ with respect to a selector Λ can be a candidate of extreme points of $\widetilde{\mathcal{P}^H}$, i.e., only M^{p^Λ} can be a candidate of extreme points of $\mathcal{P}^H$.

Further, for any selector $\Lambda \in \Lambda^N$, the number of carriers of $\widetilde{M^{p^\Lambda}}$ equals $2^n - 1$, and vectors corresponding to different selectors have different carriers. From this, it is concluded that the matrices $M^{p^\Lambda}, \Lambda \in \Lambda^N$ are all extreme points of $\mathcal{P}^H$. $\qquad\square$

The *selector value* corresponding to Λ is the value m^Λ given by

$$m^\Lambda(v) = M^{p^\Lambda} \cdot \Delta^v, \quad \text{for all games } \langle N, v \rangle.$$

The *selectope* $S(v)$ for a game $\langle N, v \rangle$ is defined by

$$S(v) = \text{Conv}\{m^\Lambda(v) \mid \Lambda \in \Lambda^N\}.$$

By Proposition 6.5.2, the Harsanyi set of any game agrees with the selectope. The latter concept has been discussed recently by Derks *et al.* [16] and introduced much earlier already by Hammer *et al.* [39] in a different way. The Harsanyi set is convex due to the convexity of $\mathcal{P}^H$. Its extreme points are uniquely determined by the extreme points of $\mathcal{P}^H$. From this, it seems to be that the set of extreme points for Harsanyi set (selectope) has $\prod_{s=1}^{n} s^{\binom{n}{s}}$ candidates. They are $m^\Lambda(v)$, $\Lambda \in \Lambda^N$. Whereas, we will show later on that most of them are repeated.

A game-theoretic characterization of the extreme points of the Harsanyi set was first given in Vasil'ev [92], and independently, in Derks *et al.* [16]. They have shown that the Harsanyi set has a core-type structure.

Two games $\langle N, v \rangle$ and $\langle N, w \rangle$ are *disjoint* (see [90]) if $\Delta^v(S) \cdot \Delta^w(S) = 0$ for all $S \in \Omega$. Given a game $\langle N, v \rangle$, we define the two totally positive games v^+ and v^- by

$$v^+ = \sum_{S \mid \Delta^v(S) > 0} \Delta^v(S) u_S, \quad v^- = \sum_{S \mid \Delta^v(S) < 0} -\Delta^v(S) u_S,$$

where the sum over the empty set is defined to be the zero game v^0 with every worth $v^0(S) = 0$, $S \in \Omega$. Obviously, the games $\langle N, v^+ \rangle$ and $\langle N, v^- \rangle$ are disjoint, moreover,

$$v = v^+ - v^-.$$

From the convexity of the unanimity games if follows that every totally positive game, a nonnegative linear combination of convex unanimity games, is convex as well. The corresponding *Harsanyi mingame* $\langle N, v_H \rangle$ of $\langle N, v \rangle$ is defined by

$$
\begin{aligned}
v_H(S) &= v^+(S) - (v^-)^*(S) \\
&= v(S) + v^-(S) - [v^-(N) - v^-(N \setminus S)], \quad \text{for all } S \in \Omega.
\end{aligned}
$$

The dual game $\langle N, (v^-)^* \rangle$ is concave since $\langle N, v^- \rangle$ is a totally positive game. So $\langle N, -(v^-)^* \rangle$ is convex. And hence the Harsanyi mingame $\langle N, v_H \rangle$ is the sum of two convex games. It is always convex for every game $\langle N, v \rangle$.

Theorem 6.5.3 (Derks *et al.* [16], Vasil'ev and van der Laan [95]). *The Harsanyi set of a game $\langle N, v \rangle$ equals the core of its Harsanyi mingame $\langle N, v_H \rangle$.*

Since the Harsanyi mingame $\langle N, v_H \rangle$ is convex, the core $C(v_H)$ agrees with the Weber set $W(v_H)$. Therefore, the Harsanyi set $H(v)$ of the game $\langle N, v \rangle$ equals the Weber set $W(v_H)$ of the Harsanyi mingame $\langle N, v_H \rangle$. So, the set of extreme points of the Harsanyi set $H(v) = \{m^\pi(v_H) \mid \pi \in \Pi^N\}$, is the set of marginal contribution vectors of the Harsanyi mingame. Therefore, there are at most $n!$ elements. Now we present a formula for calculating the extreme point $ex^\pi(H(v))$, $\pi \in \Pi^N$, of the Harsanyi set $H(v)$, *i.e.*, the marginal contribution vector $m^\pi(v_H)$, $\pi \in \Pi^N$, of the Harsanyi mingame $\langle N, v_H \rangle$.

Theorem 6.5.4 (Derks *et al.* [16]). *For a game $\langle N, v \rangle$ and any permutation $\pi \in \Pi^N$, the corresponding extreme point $ex^\pi(H(v))$ of the Harsanyi set $H(v)$ is given by*

$$ex_i^\pi(H(v)) = \sum_{T \subseteq \pi^i \setminus \{i\}} \Delta^{v^+}(T \cup \{i\}) - \sum_{T \subseteq N \setminus \pi^i} \Delta^{v^-}(T \cup \{i\}), \quad \text{for all } i \in N.$$

Proof. Since for any $\pi \in \Pi^N$, $ex^\pi(H(v)) = m^\pi(v_H)$, together with the linearity of the marginal contribution and $v_H = v^+ - (v^-)^*$, we have, for any player $i \in N$,

$$ex_i^\pi(H(v)) = m_i^\pi(v_H) = m_i^\pi(v^+) - m_i^\pi((v^-)^*)$$

$$= v^+(\pi^i) - v^+(\pi^i \setminus \{i\}) - \left[(v^-)^*(\pi^i) - (v^-)^*(\pi^i \setminus \{i\})\right]$$

$$= \sum_{T \subseteq \pi^i} \Delta^{v^+}(T) - \sum_{T \subseteq \pi^i \setminus \{i\}} \Delta^{v^+}(T) - \left[v^-\left(N \setminus (\pi^i \setminus \{i\})\right) - v^-(N \setminus \pi^i)\right]$$

$$= \sum_{T \subseteq \pi^i \setminus \{i\}} \Delta^{v^+}(T \cup \{i\}) - \left[\sum_{T \subseteq N \setminus (\pi^i \setminus \{i\})} \Delta^{v^-}(T) - \sum_{T \subseteq N \setminus \pi^i} \Delta^{v^-}(T)\right]$$

$$= \sum_{T \subseteq \pi^i \setminus \{i\}} \Delta^{v^+}(T \cup \{i\}) - \sum_{T \subseteq N \setminus \pi^i} \Delta^{v^-}(T \cup \{i\}). \qquad \square$$

By Theorem 6.5.4, we present a recursive algorithm for computing the set of extreme points of the Harsanyi set of any game $\langle N, v \rangle$, which is based on constructing greedily two sequences of reduced games related to positive dividends and negative dividends, respectively.

Recursive algorithm.

- Step 1. For a game $\langle N, v \rangle$, input its column vector v. Calculate its dividend vector $\Delta^v = M^\Delta \cdot v$, and derive the two dividend vectors Δ^{v^+}, Δ^{v^-};

- Step 2. Generate a permutation $\pi \in \Pi^N$, let $ex_i^\pi(H(v)) = 0$ for all $i \in N$. Take $i = 1$, $N_1 = N$, and $\Delta^{v^{1+}} = \Delta^{v^+}$, $\Delta^{v^{1-}} = \Delta^{v^-}$;

- Step 3. Do
$$ex_{\pi(i)}^\pi(H(v)) = \Delta^{v^{i+}}(\pi(i)) - \sum_{T \ni \pi(i)} \Delta^{v^{i-}}(T).$$

- Step 4. Let $i = i + 1$ and $N_i = N_i \setminus \{\pi(i)\}$.

  If $i \leq n$, construct two reduced games $\langle N_i, v^{i+} \rangle$ and $\langle N_i, v^{i-} \rangle$ by
$$\Delta^{v^{i+}}(S) = \Delta^{v^{(i-1)+}}(S) + \Delta^{v^{(i-1)+}}(S \cup \{\pi(i-1)\}),$$
$$\Delta^{v^{i-}}(S) = \Delta^{v^{(i-1)-}}(S), \qquad\qquad \text{for all } S \subseteq N_i.$$

  Return to Step 3.

  Else, save the vector $ex^\pi(H(v))$ and then return to Step 2.

From the point of view of the extreme allocated case, one can argue that this allocation is far from profitable. By the formula in Theorem 6.5.4, the players thereafter receive, one by one, the positive dividends of only those coalitions where they are the last player, and the negative dividends of those coalitions where they are the first player. View the procedure of the sequences of reduced games: every player $\pi(i)$ in the queue π receives all negative dividends $\Delta^{v^{i-}}(T)$ of the coalitions T he is a member of, and his own dividend $\Delta^{v^{i+}}(\pi(i))$. The resulting vector is called a *greedy allocation* by Derks *et al.* in [16], because looking in the reverse direction of the queue a player may grab all positive dividends yet to come while leaving the negative dividends for the players lower in the queue.

6.6   The extreme points of the Weber set

As we discussed in Remark 6.3.10, it is pivotal to study the set of extreme points of $\mathcal{P}^W$ as well as of $\mathcal{Q}^W$ for characterizing the Weber set. Recall that an extreme point of a linear system $X(A, b)$ can be recognized by its carrier and the technique we have used for the set of extreme points of the Harsanyi set. A linear system associated with $\mathcal{P}^W$ and $\mathcal{Q}^W$ is constructed in the following, and their extreme points are studied in terminology of the carrier.

Let $M^p \in \mathcal{P}^W$, $M^q = M^p M^\Delta \in \mathcal{Q}^W$. Since $[M^p]_{i,S} = 0$ and $[M^q]_{i,S} = -[M^q]_{i,S \cup \{i\}}$, for all $S \in \Omega, i \notin S$, it is sufficient and necessary for M^p, M^q to know all of these $[M^p]_{i,S}$ and $[M^q]_{i,S}$ such that $S \in \Omega, i \in S$. We construct the following linear system:

$$
\begin{cases}
\sum_{i \in S} \left[M^p \right]_{i,S} = 1, & \text{for all } S \in \Omega; \\
\left[M^p \right]_{i,S} - \sum_{T \supseteq S} \left[M^q \right]_{i,T} = 0, & \text{for all } S \in \Omega, i \in S; \\
\left[M^p \right]_{i,S} \geq 0, \ \left[M^q \right]_{i,S} \geq 0, & \text{for all } S \in \Omega, i \in S.
\end{cases}
$$

For simplicity, we denote by $\widetilde{p}, \widetilde{q}, \widetilde{p, q}$ the vectors of all entries of $[M^p]_{i,S}, [M^q]_{i,S}$, $[M^p]_{i,S}$ and $[M^q]_{i,S}$, $S \in \Omega, i \in S$ indexed in lexicographic order firstly by $S \in \Omega$ and secondly by $i \in S$ respectively, and let $\widehat{P, Q}$ be the solution set of the above linear system. From the definition of $\mathcal{P}^W$ and Proposition 6.3.4, it is concluded that

$$
\widetilde{P, Q} = \{ \widetilde{p, q} \mid M^p \in \mathcal{P}^W, M^q = M^p M^\Delta \in \mathcal{Q}^W \}.
$$

Therefore, $\widetilde{P, Q}$ is convex bounded.

Theorem 6.6.1. *The set of extreme points of $\widetilde{P, Q}$ is $\{ \widetilde{p^\pi, q^\pi} \mid \pi \in \Pi^N \}$.*

Proof. From Lemma 6.3.2 and Proposition 6.3.4, we know that $\widetilde{p^\pi, q^\pi}, \pi \in \Pi^N$ are elements of $\widetilde{P, Q}$. In the following, we show that these elements are the only candidates for being extreme points in $\widetilde{P, Q}$. By Lemma 6.5.1, this is true if we can prove that for each $\widetilde{p, q} \in \widetilde{P, Q}$, there is a permutation π such that $K(\widetilde{p^\pi, q^\pi}) \subseteq K(\widetilde{p, q})$.

Let $\widetilde{p, q} \in \widetilde{P, Q}$ and M^p, M^q be the extension of $\widetilde{p, q}$ such that $M^p \in \mathcal{P}^W, M^q \in \mathcal{Q}^W$. Since $\mathbf{1}'_N \cdot M^q = (0, 0, \cdots, 0, 1)$, there exists $i_1 \in N$ satisfying

$[M^q]_{i_1,N} > 0$, and by $[M^q]_{i_1,N\setminus\{i_1\}} = -[M^q]_{i_1,N} < 0$, it follows that there exists $i_2 \in N \setminus \{i_1\}$ satisfying $[M^q]_{i_2,N_1\setminus\{i_1\}} > 0$. This procedure can be continued up to the coalition with a single player. We get a sequence of players $i_1, i_2, \cdots, i_n$ satisfying $[M^q]_{i_1,N} > 0$ and $[M^q]_{i_r,N\setminus\{i_1,i_2,\cdots,i_{r-1}\}} > 0$, for $r = 2, 3, \cdots, n$. Denote $N_1 = N$ and $N_r = N \setminus \{i_1, i_2, \cdots, i_{r-1}\}$, for $r = 2, 3, \cdots, n$. From $[M^p]_{i,S} = \sum_{T \supseteq S} [M^q]_{i,T}$ and monotonicity of M^p, it can be deduced that $[M^p]_{i_r,N_r} > 0$, for $r = 1, 2, \cdots, n$, and $[M^p]_{i_r,R} > 0$, for all $R \ni i_r, R \subseteq N_r$.

Let $\pi = (i_n, i_{n-1}, \cdots, i_1)$ be the permutation of N. Then

$$\pi^{i_r} = \{i_n, i_{n-1}, \cdots, i_r\} = N_r.$$

According to (6.3.1), we have $[M^{q^\pi}]_{i,S} = 1$ if and only if $i = i_r$ and $S = N_r$ for $r = 1, 2, \cdots, n$, and $[M^{q^\pi}]_{i,S} = 0$ otherwise. By Lemma 6.3.2, we know that $[M^{p^\pi}]_{i,S} = 1$ if and only if $i = i_r$ and $S \ni i_r, S \subseteq N_r$ for $r = 1, 2, \cdots, n$, and $[M^{p^\pi}]_{i,S} = 0$ otherwise. Therefore, it holds that

$$K(\widetilde{p^\pi, q^\pi}) \subseteq K(\widetilde{p, q}).$$

By Lemma 6.5.1, only $\widetilde{p^\pi, q^\pi}$ with respect to a permutation π can be a candidate to be extreme point of $\widetilde{P, Q}$. Since for any permutation $\pi \in \Pi^N$, the number of carriers of $\widetilde{p^\pi, q^\pi}$ equals $n + 2^n - 1$, and elements corresponding to different permutations have different carriers, it is concluded that the elements $\widetilde{p^\pi, q^\pi}$, $\pi \in \Pi^N$ are all extreme points. $\square$

Alternative proof of Lemma 6.3.5. Let

$$\widetilde{P} = \{\widetilde{p} \mid M^p \in \mathcal{P}^W\}, \quad \widetilde{Q} = \{\widetilde{q} \mid M^q \in \mathcal{Q}^W\}.$$

Then $\widetilde{P}, \widetilde{Q}$ are the projections of $\widetilde{P, Q}$ on $\mathcal{P}^W$ and $\mathcal{Q}^W$ respectively. They are convex and so, $Ex(\widetilde{P}), Ex(\widetilde{Q})$ should be the projections of $Ex(\widetilde{P, Q})$ on $\mathcal{P}^W$ and $\mathcal{Q}^W$ respectively. By Theorem 6.6.1, we have

$$Ex(\widetilde{P}) = \{\widetilde{p^\pi} \mid \pi \in \Pi^N\}, \quad Ex(\widetilde{Q}) = \{\widetilde{q^\pi} \mid \pi \in \Pi^N\}.$$

So for any $M^p \in \mathcal{P}^W, M^q = M^p M^\Delta \in \mathcal{Q}^W$ and the corresponding vectors $\widetilde{p} \in \widetilde{P}, \widetilde{q} \in \widetilde{Q}$, there are nonnegative weights $\lambda_\pi, \pi \in \Pi^N$, satisfying $\sum_{\pi \in \Pi^N} \lambda_\pi = 1$,

such that

$$\widetilde{p} = \sum_{\pi \in \Pi^N} \lambda_\pi \widetilde{p^\pi} \text{ and } \widetilde{q} = \sum_{\pi \in \Pi^N} \lambda_\pi \widetilde{q^\pi}.$$

Since $[M^p]_{i,S} = 0$ and $[M^q]_{i,S} = -[M^q]_{i,S\cup\{i\}}$, for all $S \in \Omega, i \notin S$, therefore,

$$M^p = \sum_{\pi \in \Pi^N} \lambda_\pi M^{p^\pi} \text{ and } M^q = \sum_{\pi \in \Pi^N} \lambda_\pi M^{q^\pi}.$$

i.e., $Ex(\mathcal{P}^W) = \{M^{p^\pi} \mid \pi \in \Pi^N\}$, and $Ex(\mathcal{Q}^W) = \{M^{q^\pi} \mid \pi \in \Pi^N\}$.   □

Remark 6.6.2. *Derks* [18] *applied a similar approach to study the extreme points of the collection* Q^W *associated with the Weber set of all systems (vectors)* $q = [q_i^S]_{i\in S}^{S \in \Omega}$ *such that*

$$q(N) = \sum_{i \in N} q_i^N = 1; \quad q_i^S \geq 0, \sum_{i \in S} q_i^S = \sum_{j \in N \backslash S} q_j^{S \cup \{j\}}, \quad \text{for all } S \in \Omega, i \in S.$$

As we mentioned in Remark 6.2.4, *these equalities are used to access the procedure for generating the sequence of players in the permutation* π. *In our approach, both* $\mathcal{P}^W$ *and* $\mathcal{Q}^W$ *are considered in one linear system. The non-member weights* $[M^q]_{i,S\backslash\{i\}}$, *for all* $S \in \Omega, i \in S, S \neq \{i\}$, *and the equalities*

$$[M^q]_{i,S} = -[M^q]_{i,S\backslash\{i\}}, \ S \neq \{i\}, \quad \sum_{T \supseteq S} [M^q]_{i,T} \geq 0, \ \text{for all } S \in \Omega, i \in S,$$

yield a clearer procedure for generating the sequence of players in the permutation π.

6.7   Some related matrix representations

We list some matrix representations involved in Chapters 2 and 6. Table 6.7.1 is about three bases and their relations in terms of the Moebius transformation and its complementarity on the game space, the definitions and results can be found in Chapter 2. Table 6.7.2 presents the Harsanyi payoff vectors with respect to different sharing systems according to the objects distributed. Table 6.7.3 lists different representations for the Harsanyi set and the Weber set in terms of three collections of different types of sharing matrices and the extreme points of these collections.

Basis	Games	Transformation	Representation
$\{\langle N, e_S \rangle \mid S \in \Omega\}$	unity		
$\{\langle N, u_S \rangle \mid S \in \Omega\}$	unanimity	Moebius	$e_S = M^\Delta \cdot u_S$
$\{\langle N, \bar{u}_S \rangle \mid S \in \Omega\}$	complementary unanimity	complementary Moebius	$e_S = M^{\bar{\Delta}} \cdot \bar{u}_S$

<div align="center">

Table 6.7.1: Bases of the game space and the corresponding linear transformations

</div>

Vector	Name	Sharing matrix	$\Phi^p(v)/m^\pi(v)$
v	worth	M^q	$\Phi^p(v) = M^q \cdot v$
		M^{q^π}	$m^\pi(v) = M^{q^\pi} \cdot v$
$\Delta^v = M^\Delta \cdot v$	dividend	M^p	$\Phi^p(v) = M^p \cdot \Delta^v$
		M^{p^π}	$m^\pi(v) = M^{p^\pi} \cdot \Delta^v$
$\bar{\Delta}^v = M^{\bar{\Delta}} \cdot v$	complementary dividend	$M^{\bar{p}}$	$\Phi^p(v) = M^{\bar{p}} \cdot \bar{\Delta}^v$
		$M^{\bar{p}^\pi}$	$m^\pi(v) = M^{\bar{p}^\pi} \cdot \bar{\Delta}^v$

<div align="center">

Table 6.7.2: Different sharing matrices for the Harsanyi payoff vectors and marginal vectors

</div>

Solution	Representation	Extreme point/Relationship
$H(v) =$	$\{M^p \cdot \Delta^v \mid M^p \in \mathcal{P}^H\}$	$Ex(\mathcal{P}^H) = \{M^{p^\Lambda} \mid \Lambda \in \Lambda^N\}$
	$\{M^q \cdot v \mid M^q \in \mathcal{Q}^H\}$	$\mathcal{Q}^H = \mathcal{P}^H M^\Delta$
	$\{M^{\bar{p}} \cdot \bar{\Delta}^v \mid M^{\bar{p}} \in \bar{\mathcal{P}}^H\}$	$\bar{\mathcal{P}}^H = \mathcal{P}^H M^{\bar{\Delta}}$
$W(v) =$	$\{M^p \cdot \Delta^v \mid M^p \in \mathcal{P}^W\}$	$Ex(\mathcal{P}^W) = \{M^{p^\pi} \mid \pi \in \Pi^N\}$
	$\{M^q \cdot v \mid M^q \in \mathcal{Q}^W\}$	$\mathcal{Q}^W = \mathcal{P}^W M^\Delta$
	$\{M^{\bar{p}} \cdot \bar{\Delta}^v \mid M^{\bar{p}} \in \bar{\mathcal{P}}^W\}$	$\bar{\mathcal{P}}^W = \mathcal{P}^W M^{\bar{\Delta}}$

<div align="center">

Table 6.7.3: Different representations for the Harsanyi set and the Weber set

</div>

Conclusion

In this monograph, the algebraic representation and the matrix approach are applied to study linear operators on the game space, more precisely, linear transformations on games and linear values. In terms of the basic notion of a coalitional matrix, these linear operators are represented algebraically by products of the corresponding coalitional matrix and the worth vector. We preform a matrix analysis in the setting of cooperative game theory, to study axiomatizations of linear values, by investigating appropriate properties of these representation matrices. Particularly, the Shapley value is the most important representative. In summary, the concepts of eigenvalues, eigenvectors, null space, the diagonalization procedure and the similarity property for matrices, the system of linear equations and its solution set, the Moebius transformation and the complementary Moebius transformation, the basis for a linear space and so on (see Table 4.5.1-4.5.3 and Table 6.7.1-6.7.3), can be applied successfully to cooperative game theory. We conclude that the matrix analysis is a new and powerful technique for research in the field of cooperative game theory.

This monograph is an initialization for using systematically the algebraic representation and the matrix approach in the research field of cooperative game theory. There are still many more open problems. Recall some other well-studied game-theoretic models. Cooperative games with coalition structure were initiated by Aumann and Drèze [3], then studied by Owen [68]. Later this approach was extended in Winter [100] to games with level structure. Another model of a game with limited cooperation, presented by means

of communication graphs, was introduced in Myerson [64]. The generalization of the Owen and the Myerson values, applied to the combination of both models and resulting in TU-games with both independent coalition and cooperation structures, was investigated by Vázquez-Brage, García-Jurado, and Carreras [97], and values for graph-restricted games with coalition structure by Khmelnitskaya [51].

Multi-choice cooperative games introduced by Hsiao and Raghavan [46,47] and van den Nouweland *et al.* [89] are natural extensions of traditional cooperative games. Whereas in a traditional cooperative game each player may have only two options concerning cooperation, being either active or inactive, in a multi-choice context each player may have additional participation opportunities on a finite set of activity levels. Results on multi-choice games can be also found in Calvo and Santos [10], Calvo, Gutiérrez and Santos [11], Klijn, Slikker and Zarzuelo [53], Peters and Zank [70]. Additionally, the reader can look at the survey on multi-choice cooperative games in Branzei, Dimitrov, and Tijs [9]. Another type of generalized cooperative games in characteristic function form, called set games, was introduced by Hoede [45]. The worth of a coalition for a set game is expressed by a set instead of by a real number as for TU-games. A *set game*, is a triple $\langle N, v, \mathcal{U} \rangle$, where N is a finite player set, $\mathcal{U}$ denotes an abstract set, called *universe*, v is a mapping $v \colon 2^N \to 2^{\mathcal{U}}$ satisfying $v(\emptyset) = \emptyset$. The worth $v(S)$ of a coalition S is a subset of the universe $\mathcal{U}$. For this class of games, the notion of values, being a solution concept similar to those for TU-games, can be found in Sun [79], Sun and Xu [82], Sun and Driessen [81], Sun *et al.* [80,83].

In terms of the algebraic approach, we present another game model with respect to an Abelian group as follows. A *group game*, is a triple $\langle N, v, \mathcal{F} \rangle$, where N is a finite player set, $\mathcal{F}$ denotes an Abelian group with operation $\oplus$, v is a mapping $v \colon 2^N \to \mathcal{F}$ satisfying $v(\emptyset) = e$, where e is the unit of group $\mathcal{F}$. So the worth $v(S)$ of a coalition S is an element of the group $\mathcal{F}$. For any group game $\langle N, v, \mathcal{F} \rangle$, the Shapley value $Sh(N, v, \mathcal{F})$ can also be defined as

$$Sh_i(N, v, \mathcal{F}) = \bigoplus_{\substack{S \subseteq N \\ S \ni i}} \left[v(S) \oplus v^{-1}(S \setminus \{i\}) \right]^{\frac{(s-1)!(n-s)!}{n!}}, \quad \text{for all } i \in N,$$

where $[v(T)]^{\frac{a}{b}} = v^a(T) \oplus [v^b(T)]^{-1}$, for all $T \in \Omega$ and $a, b \in \mathbb{N}$.

The main question is how to use the matrix approach or the generalized algebraic approach for studying these models of cooperative games and the corresponding values. It looks like that there exists indeed a possibility for modelling. For instance, cooperative games with limited cooperation and their values may be modelled as coalitional matrices with some special blocks. Continuing what we started in this monograph, to investigate these algebraic models and corresponding solution concepts in cooperative game theory will be an important part of our future work.

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Index

159

Acknowledgements

Working towards a Ph.D. thesis is a long road with many bends and turns. It is impossible to do on one's own, and I had great help and useful advice from a lot of people. In the end, it is this help that kept me going on this road, and made this trip as the trip of my life. I am grateful to all, and there are many people which I may forget to mention here, although not intentionally.

First and foremost, I want to thank Prof. Dr. Kees Hoede and Prof. Dr. Xueliang Li for offering me this opportunity to be a Ph.D. student of the University of Twente, to thank Prof. Dr. Marc Uetz and Prof. Dr. Xueliang Li for being my promoters. Especially, when I discussed with Kees Hoede, he always provided me some new ideas and meritorious suggestions.

Special thanks to my supervisor Dr. Theo Driessen. This thesis could not have been written without his stimulation, instruction and cooperation. I have had many interesting and fruitful discussions with him during this period of four years. His profound knowledge of cooperative game theory and scrupulous scientific spirit has been very helpful in the past and will be useful too in my further research. His wife Dr. Anna Khmelnitskaya has friendly made beneficial remarks to improve the manuscript of my thesis. I would like to thank them not only for their contributions to this thesis, but also for their warmhearted caring for my life in the Netherlands.

I would like to thank all the members of my graduation committee for participating as there are Prof. Dr. C. Hoede, Prof. Dr. A.J. Mouthaan, Prof. Dr. H.J.M. Peters, Prof. Dr. S.H. Tijs, Dr. W. Kern, Dr. A. Khmelnitskaya and Dr. H. Sun.

I enjoyed the pleasant working environment at the group of Discrete Mathematics and Mathematical Programming at University of Twente. I want to thank everyone from DMMP for this, and in addition I am grateful to Georg Still, Dini Heres-Ticheler, Harry Aarts and Johann Hurink for their help. I greatly appreciate Xinhui Wang's help for my work and personal life in the Netherlands. Thanks also go to my colleagues and friends from Northwestern Polytechnical University, namely Xiaodong Liu, Hao Sun, Ligong Wang, Shenggui Zhang and Haixing Zhao, for advising me well to go ahead despite difficulties.

A lot of thanks to my friends met in Enschede: Wei Bai, Wei Cheng, Hongxi Guo, Yunkai Guo, Jie Hong, XiuLi Li, Tao Li, Jiwu Lu, Tingting Luan, Chunlin Song & Huaqiong Yang, Qiang Tang & Shenglan Hu, Tao Yang, Mingshang Yu, Yang Zhang. You are great friends for bringing me lots of laughter and happiness.

Finally, I would deeply thank my parents, parents-in-law, brothers and sisters. They provided me with generous support and encouragement during these years, even though they did not always understand what I was doing. Most of all, I gratefully acknowledge my wife Jun Su for her love, support, and the best gift I have ever received from her, my son Anzhe Xu.

Genjiu Xu
July 2008, Enschede

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Genjiu Xu was born on November 5, 1978, in Lujiang County of Anhui Province, P.R. China. From 1984 until 1997 he attended primary and secondary school in his hometown. In September 1997, he started to study applied mathematics at Northwestern Polytechnical University in Xi'an. After receiving his Bachelor degree in July 2001, he continued his mathematics study at the same university and became a graduate student. He specialized in operations research, in particular, graph theory, and subsequently, he completed his Master's thesis, entitled 'Contributions to Digraphs', under the supervision of Prof. Dr. Xueliang Li. In April 2004, he graduated with honors and received his Master degree. Afterwards, he has been employed as a teacher affiliated with the School of Science at Northwestern Polytechnical University. Since then, he has been teaching mathematics as well as performing scientific research in the fields of game theory and graph theory.

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